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# MODIFICATIONS AND COHOMOLOGIES OF SOLVMANIFOLDS

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Abstract. Let $G/\Gamma$ be a compact solvmanifold. In this paper, unifying approaches of previous works on the subject, we develop tools for the computation of the de Rham (and Dolbeault) cohomology of various classes of compact (complex) solvmanifolds. In particular, we generalize the modification method of Witte to any subtorus of the maximal torus of the Zariski closure $\mathcal{A}(\mathrm{Ad}_G(G))$ of $\mathrm{Ad}_G(G)$, extending it also to the Dolbeault cohomology case.

## 1. Introduction

Let $G/\Gamma$ be a compact quotient of a connected and simply connected solvable Lie group $G$ by a lattice $\Gamma$, that we shall call a *compact solvmanifold*.

Unlike compact nilmanifolds, de Rham and Dolbeault cohomologies of compact solvmanifolds $G/\Gamma$ are not in general isomorphic to the corresponding cohomologies of the Lie algebra $\mathfrak{g}$ of $G$.

Special cases when the de Rham cohomology is given the invariant one are provided by compact solvmanifolds of completely solvable Lie groups [14] and, more generally, by compact solvmanifolds for which the *Mostow condition* holds, i.e. for which the algebraic closure $\mathcal{A}(\mathrm{Ad}_G(G))$ of $\mathrm{Ad}_G(G)$ coincides with the one $\mathcal{A}(\mathrm{Ad}_G(\Gamma))$ of $\mathrm{Ad}_G(\Gamma)$ (see [19] and [21, Corollary 7.29]).

In the general case, techniques for the computation of the de Rham cohomology of compact solvmanifolds were provided by the third author in [16, 17], constructing finite dimensional cochain complexes which compute the de Rham cohomology.

On the other hand, the first two authors in [6], using results by D. Witte [25] on the superigidity of lattices in solvable Lie groups, obtained a different proof of a result of Guan [13], which can be applied to compute the Betti numbers of a compact solvmanifold $G/\Gamma$, even in the case that the solvable Lie group $G$ and the lattice $\Gamma$ do not satisfy the Mostow condition. The basic idea of this method is to modify the solvable Lie group $G$ into a new solvable Lie group $\tilde{G}$ (diffeomorphic to $G$) and possibly considering a finite index subgroup $\tilde{\Gamma}$ of $\Gamma$, in such a way that the new compact solvmanifold $\tilde{G}/\tilde{\Gamma}$ satisfies the Mostow condition (cf. Theorem 2.5 here).

As a first step, in Section 3 we try to unify these two approaches for the computation of the de Rham cohomology of $G/\Gamma$. Let $\mathfrak{n}$ be the nilradical of the Lie algebra $\mathfrak{g}$ of $G$. By [11, Proposition III.1.1] it turns out that $\mathfrak{g}$ decomposes as direct sum of vector spaces as $V \oplus \mathfrak{n}$ with $V \cong \mathbb{R}^k$ vector space such that $\mathrm{ad}(A)_s(B) = 0$, for any $A, B \in V$, where $\mathrm{ad}(A)_s$ denotes the semisimple part of $\mathrm{ad}(A)$. Using a natural map $\mathrm{Ad}_s : G \to \mathrm{Aut}(\mathfrak{g})$, defined as the extension of

$$\mathrm{ad}_s : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g}), A + n \in \mathfrak{g} = V + \mathfrak{n} \mapsto (\mathrm{ad}_A)_s(X)$$

to $G$, we describe a general modification with respect to an algebraic subtorus $S$ of the maximal torus $T = \mathcal{A}(\mathrm{Ad}_s(G))$ of $\mathcal{A}(\mathrm{Ad}_G(G))$, which extends the Witte modification (cf. Proposition 3.2).

The relation between the $S$-modification and de Rham cohomology is provided by Theorem 3.5, where we show a condition on the lattice $\Gamma$ under which the de Rham cohomology $H^*_{dR}(G/\Gamma)$

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is isomorphic to the Chevalley-Eilenberg cohomology $H^*(\mathfrak{g}^S)$ of the Lie algebra $\mathfrak{g}^S$ of the $S$-modification $G^S$. This result extends both the ones in [16, 17] and the modification method in [6] (see Remark 3.6 and Example 3.7).

Next, we turn to compact solvmanifolds $G/\Gamma$ endowed with an invariant complex structure $J$ (i.e., with a complex structure induced by a left invariant one on $G$). We will call $(G/\Gamma, J)$ a *complex solvmanifold*. The aim is to compute the Dolbeault cohomology $H^{*,*}_{\bar\partial}(G/\Gamma)$.

First we show that if $G/\Gamma$ has an invariant complex structure $J$ which commutes with $\mathrm{Ad}_s$, then $J$ induces a left-invariant complex structure on the $S$-modification $G^S$ and under a further assumption on $\Gamma$ we prove that the new complex solvmanifold $(G^S/\Gamma, J)$ is biholomorphic to $(G/\Gamma, J)$ (Proposition 3.8).

Every compact solvmanifold $G/\Gamma$ is a fiber bundle (called the *Mostow fibration*)

$$N/\Gamma\cap N = N\Gamma/\Gamma \longrightarrow G/\Gamma \longrightarrow G/N\Gamma = (G/N)/(\Gamma/\Gamma\cap N)$$

over a torus with a compact nilmanifold $N/\Gamma\cap N$ as fiber, where $N$ is the niradical of $G$. In Section 4 we show that the Mostow bundle of a compact solvmanifold endowed with an invariant complex structure preserving the nilradical is holomorphic if and if only the complex structure $J_N$ on the nilradical $N$ is $C$-invariant, where $C$ is a simply connected nilpotent subgroup $C\subset G$ such that $G = C\cdot N$ as a product. We also study this property in relation to modifications (cf. Corollary 4.3).

Steps in the direction of the computation of the Dolbeault cohomology of a complex solvmanifold $(G/\Gamma, J)$ were previously taken by the third author in [16, 17]. In particular, he constructed a finite dimensional cochain complex which allows to compute the Dolbeault cohomology of complex parallelizable solvmanifolds (i.e. compact quotients of complex solvable Lie groups).

In Section 5 we generalize these results constructing a differential bigraded algebra (DBA) which allows to compute the Dolbeault cohomology in more general situations, provided the Mostow fibration is holomorphic and $J$ is $\mathrm{Ad}_s$-invariant (Theorem 5.2).

This kind of complex allows to compute the Dolbeault cohomology of solvmanifolds both of non splitting type (see Example 5.3) and of the ones whose solvable group is a semidirect product $G = \mathbb{C}^n \ltimes_\phi N$. The latter can be dealt in the general situation when $N$ is a simply connected nilpotent Lie group (which in general is not the nilradical of $G$) even if $\phi$ is not semisimple, provided that $G$ has a lattice $\Gamma$ which can be written as $\Gamma = \Gamma' \ltimes_\phi \Gamma''$ (such that $\Gamma'$ and $\Gamma''$ are lattices of $\mathbb{C}^n$ and $N$ respectively and for any $t\in\Gamma'$ the action $\phi(t)$ preserves $\Gamma''$) and the inclusion $\bigwedge^{*,*}\mathfrak{n}^*\subset A^{*,*}(N/\Gamma'')$ induces an isomorphism with the invariant Dolbeault cohomology (Theorem 5.5).

In Section 6 we apply these results to modifications of solvable Lie algebras.

We show that there exists a finite index subgroup $\tilde\Gamma$ of $\Gamma$ and a real solvable Lie algebra $\breve{\mathfrak{g}}$ with a complex structure $\breve J$ such that

$$H^{*,*}_{\bar\partial}(G/\tilde\Gamma)\cong H^{*,*}_{\bar\partial}(\breve{\mathfrak{g}}, \breve J)$$

(Theorem 6.4). If $G$ is completely solvable, then the modification $\breve{\mathfrak{g}}$ is also completely solvable.

If we suppose that the complex structure $J$ is abelian [2], we show (Theorem 6.6) that there exists a finite index subgroup $\tilde\Gamma$ of $\Gamma$ and a real solvable Lie algebra $\breve{\mathfrak{g}}$ with an abelian complex structure $\breve J$ such that

$$H^{*,*}_{\bar\partial}(G/\tilde\Gamma)\cong H^*(\breve{\mathfrak{g}}^{1,0})\otimes\wedge^*\breve{\mathfrak{g}}^{0,1}\,.$$

Finally we prove (Corollary 6.8) that if $(G/\Gamma, J)$ is a complex parallelizable solvmanifold, then there exists a finite index subgroup $\tilde\Gamma$ of $\Gamma$ and a complex solvable Lie algebra $\breve{\mathfrak{g}}$ such that

$$H^{*,*}_{\bar\partial}(G/\tilde\Gamma)\cong \wedge^*\breve{\mathfrak{g}}^{1,0}\otimes H^*(\breve{\mathfrak{g}}^{0,1})\,.$$

## 2. Preliminaries

Compact solvmanifolds $G/\Gamma$ are determined up to diffeomorphism by their fundamental groups (which coincide with their lattices $\Gamma$). This can be formulated by the following

**Theorem 2.1.** [21, Theorem 3.6] *If $\Gamma_1$ and $\Gamma_2$ are lattices in simply connected solvable Lie groups $G_1$ and $G_2$, respectively, and $\Gamma_1$ is isomorphic to $\Gamma_2$, then $G_1/\Gamma_1$ is diffeomorphic to $G_2/\Gamma_2$.*

Let $\mathfrak{g}$ be a solvable Lie algebra and $\mathfrak{n}$ the nilradical of $\mathfrak{g}$. By Proposition III.1.1 in [11] there exists a vector space $V \cong \mathbb{R}^k$ such that $\mathfrak{g} = V \oplus \mathfrak{n}$ as direct sum of vector spaces and $\mathrm{ad}(A)_s(B) = 0$, for any $A, B \in V$, where $\mathrm{ad}(A)_s$ denotes the semisimple part of $\mathrm{ad}(A)$.

Like in [16] one can define the map

$$\mathrm{ad}_s : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$$

by $\mathrm{ad}_s(A + X)(Y) = (\mathrm{ad}_A)_s(Y)$, for $A \in V$ and $X \in \mathfrak{n}$ and $Y \in \mathfrak{g}$.

Therefore, $\mathrm{ad}_s$ is linear and

$$[\mathrm{ad}_s(\mathfrak{g}), \mathrm{ad}_s(\mathfrak{g})] = 0.$$

Since the commutator $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$ is contained in the nilradical $\mathfrak{n}$ of $\mathfrak{g}$, the map $\mathrm{ad}_s : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ is a representation of $\mathfrak{g}$ and the image $\mathrm{ad}_s(\mathfrak{g})$ is abelian and consists of semisimple elements.

Note that, since $\mathrm{ad}(A)_s(B) = 0$ for any $A, B \in V$, the vector space $V$ is a trivial sub-module of $\mathfrak{g}$ which is complement to $\mathfrak{n}$.

We will denote by

$$\mathrm{Ad}_s : G \to \mathrm{Aut}(\mathfrak{g})$$

the extension of the map $\mathrm{ad}_s$ to $G$. By the previous properties, it follows that $\mathrm{Ad}_s(G)$ is diagonalizable.

Let $T = \mathcal{A}(\mathrm{Ad}_s(G))$ the Zariski closure of $\mathcal{A}(\mathrm{Ad}_s(G))$ in $\mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$. Then $T$ is diagonalizable and it is a torus in $\mathcal{A}(\mathrm{Ad}_G(G))$.

**Lemma 2.2.** *The Zariski closure $T = \mathcal{A}(\mathrm{Ad}_s(G))$ of $\mathrm{Ad}_s(G)$ is a maximal torus of the Zariski closure $\mathcal{A}(\mathrm{Ad}_G(G))$ of $\mathrm{Ad}_G(G)$).*

*Proof.* Since $G$ is simply connected, the map

$$\phi : V \oplus \mathfrak{n} \to G, A + X \mapsto \exp(A)\exp(X)$$

is a global diffeomorphism. Moreover, we obtain

$$\mathrm{Ad}_{\exp(A)\exp(X)} = \mathrm{Ad}_{\exp(A)}\mathrm{Ad}_{\exp(X)} = (\exp(\mathrm{ad}_s A)\exp(\mathrm{ad}_n A))\mathrm{Ad}_{\exp(X)}.$$

Therefore

$$\mathrm{Ad}_s(\exp(V)) = \mathrm{Ad}_s(G).$$

Consider $\mathrm{Ad}_{\exp(A)} \in \mathcal{A}(\mathrm{Ad}_G(G))$. By the Jordan decomposition we have

$$\mathrm{Ad}_{\exp(A)} = \mathrm{Ad}_{s\exp(A)} \exp_{\mathrm{ad}_n(A)}.$$

Thus $\mathrm{Ad}_{s\exp(A)} \in \mathcal{A}(\mathrm{Ad}_G(G))$. Let $U$ be the nilpotent radical of $\mathcal{A}(\mathrm{Ad}_G(G))$, then $\mathcal{A}(\mathrm{Ad}_G(G))$ is the semidirect product $T \ltimes U$, where $T = \mathcal{A}(\mathrm{Ad}_s(G))$. For every $A \in V$ it holds

$$\exp(\mathrm{ad}_s A) \in T, \quad \exp(\mathrm{ad}_n A) \in U$$

and for every $X \in \mathfrak{n}$, $\mathrm{Ad}(\exp(X))$ is unipotent. We can show that $\mathrm{Ad}_{\exp(A)\exp(X)} \in T \ltimes U$ and then $\mathrm{Ad}_G(G) \subset T \cdot U$ (as a product). Consequently $\mathcal{A}(\mathrm{Ad}_G(G)) = T \cdot U$ and thus $T$ is a maximal torus in $\mathcal{A}(\mathrm{Ad}_G(G))$. $\square$

Let $G$ be a simply connected solvable Lie group with a lattice $\Gamma$ and $\mathfrak{g}$ the Lie algebra of $G$. Take a basis $X_1, \dots, X_n$ of $\mathfrak{g}_{\mathbb{C}}$ such that $\mathrm{ad}_s$ is represented by diagonal matrices as

$$\mathrm{Ad}_{sg} = \mathrm{diag}\left(\alpha_1(g), \dots, \alpha_n(g)\right),$$

for any $g \in G$.

We recall the following result proved in [16] (Corollary 1.10).

**Theorem 2.3.** [16] *Let $G$ be a simply connected solvable Lie group with a lattice $\Gamma$ and $\mathfrak{g}$ the Lie algebra of $G$. Suppose that for any $\{i_1, \dots, i_p\} \subset \{1, \dots, n\}$ if the product $\alpha_{i_1 \dots i_p}$ of the characters $\alpha_{i_1}, \dots, \alpha_{\dots i_p}$ is non-trivial, then the restriction of $\alpha_{i_1 \dots i_p}|_\Gamma$ in $\Gamma$ is also non-trivial. Then the isomorphism*

$$H^*(G/\Gamma, \mathbb{C}) \cong H^*(\mathfrak{g}_{\mathbb{C}})$$

*holds.*

As a consequence, in particular we obtain the isomorphism $H^*_{dR}(M) \cong H^*(\mathfrak{g})$.

**Remark 2.4.** There exist examples of compact solvmanifolds which satisfy the assumption of the previous theorem, but $G$ and $\Gamma$ do not satisfy the Mostow condition (see for instance Example 3 in [16]).

When $G$ and $\Gamma$ do not satisfy the Mostow condition, one may apply the Witte modification [25], which is a variation of the construction of the nilshadow. In general, one has that $\mathcal{A}(\mathrm{Ad}_G(G)) = T \ltimes U$ is not unipotent, where $T$ is a non trivial maximal torus of $\mathcal{A}(\mathrm{Ad}_G(G))$ and $T = T_{split} \times T_{cpt}$, where by $T_{split}$ we denote the maximal $\mathbb{R}$-split subtorus of $T$ and by $T_{cpt}$ the maximal compact subgroup of $T$. The basic idea for the construction of the nilshadow is to kill $T$ in order to obtain a nilpotent group. In order to do this we can define, following [25], a natural homomorphism $\pi : G \to T$, which is the composition of the homomorphisms:

$$(*) \qquad \pi : G \overset{\mathrm{Ad}}{\to} \mathcal{A}(\mathrm{Ad}_G(G)) \overset{\text{projection}}{\longrightarrow} T,$$

and the map

$$\Delta : G \to T \ltimes G, g \mapsto \Delta(g) = (\pi(g)^{-1}, g).$$

The traditional nilshadow construction kills the entire maximal torus $T$ in order to get the nilpotent Lie group $\Delta(G)$. Witte introduced in [25] a variation of the nilshadow construction, killing only a subtorus $S$ of $T_{cpt}$. It is well known that, for every subtorus $S$ of a compact torus $T$, there is a torus $S^\perp$ complementary to $S$ in $T$, i.e. such that $T = S \times S^\perp$.

As a consequence of [25, Proposition 8.2] it was proved the following

**Theorem 2.5.** [6, 13] *Let $M = G/\Gamma$ be a compact solvmanifold, quotient of a simply connected solvable Lie group $G$ by a lattice $\Gamma$ and let $T_{cpt}$ be a compact torus such that*

$$T_{cpt}\mathcal{A}(\mathrm{Ad}_G(\Gamma)) = \mathcal{A}(\mathrm{Ad}_G(G)).$$

*Then there exists a subgroup $\tilde{\Gamma}$ of finite index in $\Gamma$ and a simply connected normal subgroup $\tilde{G}$ of $T_{cpt} \ltimes G$ such that*

$$\mathcal{A}(\mathrm{Ad}_{\tilde{G}}(\tilde{\Gamma})) = \mathcal{A}(\mathrm{Ad}_{\tilde{G}}(\tilde{G})).$$

*Therefore, $\tilde{G}/\tilde{\Gamma}$ is diffeomorphic to $G/\tilde{\Gamma}$ and $H^*_{dR}(G/\tilde{\Gamma}) \cong H^*(\tilde{\mathfrak{g}})$, where $\tilde{\mathfrak{g}}$ is the Lie algebra of $\tilde{G}$.*

The idea of the proof in [6] is to do a modification $\tilde{G}$ of the solvable Lie group $G$ that we will review shortly. By [21, Theorem 6.11, p. 93] it is not restrictive to suppose that $\mathcal{A}(\mathrm{Ad}_G(\Gamma))$ is connected. Otherwise one considers a finite index subgroup $\tilde{\Gamma}$ of $\Gamma$ such that $\mathcal{A}(\mathrm{Ad}_G(\tilde{\Gamma}))$ is connected. Let $T_{cpt}$ be a maximal compact torus of $\mathcal{A}(\mathrm{Ad}_G(G))$ which contains a maximal compact torus $S^\perp_{cpt}$ of $\mathcal{A}(\mathrm{Ad}_G(\tilde{\Gamma}))$. There is a natural projection from $\mathcal{A}(\mathrm{Ad}_G(G))$ to $T_{cpt}$, given by the

splitting $\mathcal{A}(\mathrm{Ad}_G(G)) = (A \times T_{cpt}) \ltimes U$, where $A$ is a maximal $\mathbb{R}$-split torus and $U$ is the unipotent radical.

Let $S_{cpt}$ be a subtorus of $T_{cpt}$ complementary to $S_{cpt}^{\perp}$ so that $T_{cpt} = S_{cpt} \times S_{cpt}^{\perp}$. Let $\sigma$ be the composition of the homomorphisms:

$$\sigma : G \xrightarrow{\mathrm{Ad}} \mathcal{A}(\mathrm{Ad}_G(G)) \xrightarrow{\text{projection}} T_{cpt} \xrightarrow{\text{projection}} S_{cpt} \xrightarrow{x \to x^{-1}} S_{cpt}.$$

One may define the nilshadow map:

$$\Delta : G \to S_{cpt} \ltimes G, g \mapsto (\sigma(g), g),$$

which is not a homomorphism (unless $S_{cpt}$ is trivial and then $\sigma$ is trivial), but with respect to the product on $G$ one has

$$\Delta(ab) = \Delta(\sigma(b^{-1})a\sigma(b))\,\Delta(b), \quad \forall a, b \in G$$

and $\Delta(\gamma g) = \gamma\Delta(g)$, for every $\gamma \in \tilde{\Gamma}, g \in G$. The nilshadow map $\Delta$ is a diffeomorphism from $G$ onto it image $\Delta(G)$ and then $\Delta(G)$ is simply connected. More explicitly, the product in $\Delta(G)$ is given by:

$$\Delta(a)\Delta(b) = (\sigma(a), a)\,(\sigma(b), b) = (\sigma(a)\sigma(b), \sigma(b^{-1})a\sigma(b)\,b),$$

for any $a, b \in G$.

By construction $\mathcal{A}(\mathrm{Ad}_G(G))$ projects trivially on $S_{cpt}$ and $\sigma(\tilde{\Gamma}) = \{e\}$. Therefore

$$\tilde{\Gamma} = \Delta(\tilde{\Gamma}) \subset \Delta(G).$$

Let $\tilde{G} = \Delta(G)$. By [25, Proposition 4.10] $S_{cpt}^{\perp}$ is a maximal compact subgroup of $\mathcal{A}(\mathrm{Ad}_{\tilde{G}}(\tilde{G}))$ and $S^{\perp} \subset \mathcal{A}(\mathrm{Ad}_{\tilde{G}}(\tilde{\Gamma}))$, therefore $\tilde{G}$ and $\tilde{\Gamma}$ satisfy the Mostow condition.

By using Theorem 2.1, $G/\tilde{\Gamma}$ is diffeomorphic to $\tilde{G}/\tilde{\Gamma}$. Since $\tilde{G}$ and $\tilde{\Gamma}$ satisfy the Mostow condition, $H^*(\tilde{G}/\tilde{\Gamma}) \cong H^*(\tilde{\mathfrak{g}})$, so $H^*(G/\tilde{\Gamma}) \cong H^*(\tilde{\mathfrak{g}})$ and the inverse $\Delta^{-1}$ of the diffeomorphism

$$\Delta : G \to \tilde{G},$$

induces a finite-to-one covering map $\Delta^* : \tilde{G}/\tilde{\Gamma} \to G/\Gamma$.

## 3. Modified solvable Lie groups and $\mathrm{Ad}_s$-invariant geometry

In this section we will define a modification of a simply connected real solvable Lie group $G$ with respect to a sub-algebraic torus $S$ of the maximal torus $T = \mathcal{A}(\mathrm{Ad}_s(G))$, which extends the Witte modification $\Delta(G) = \tilde{G}$ described in the previous Section.

### 3.1. $S$-modification for any sub-algebraic torus $S \subset T$.

**Definition 3.1.** Let $G$ be a simply connected solvable Lie group. Consider the $\mathbb{C}$-diagonalizable representation $\mathrm{Ad}_s : G \to \mathrm{Aut}(\mathfrak{g})$ and the Zariski-closure $T$ of $\mathrm{Ad}_s(G)$ in $\mathrm{Aut}(\mathfrak{g})$. Let $S \subset T$ be a sub-algebraic torus. Consider the homomorphism $\pi_S : G \xrightarrow{\pi} T \to S$ (where $\pi$ is the map in $(*)$).

Then we define the new product $\bullet_S$ on $G$ by

$$a \bullet_S b = a\pi_S(a)^{-1}(b) \tag{1}$$

for $a, b \in G$. We denote by $G^S$ the Lie group $G$ endowed with the product $\bullet_S$ and call $G^S$ the $S$-modification of $G$.

Note that, by definition, $G^S$ is still solvable. By using Definition 3.1 and the results in [1] we can show the following properties.

**Proposition 3.2.** *Let $G$ be a simply connected solvable Lie group and let $G^S$ the $S$-modification of $G$, with $S$ a sub-algebraic torus of $T$. Then*

(1) *If $S = T$, then $G^S$ is the nilshadow of $G$.*

(2) *If $S = T_{cpt}$ (resp $T_{split}$), then $G^S$ is completely solvable (resp. of (I)-type).*
(3) *Suppose $G$ has a lattice $\Gamma$. If the restriction $\pi_S|_\Gamma$ is trivial, then $\Gamma$ is also a lattice in $G^S$ and hence we have a diffeomorphism $G/\Gamma \cong G^S/\Gamma$.*
(4) *Suppose that $G$ has a lattice $\Gamma$ and that the Zariski-closure of $\mathrm{Ad}_s(\Gamma)$ is connected. Take $S$ a subtorus complementary to a maximal compact torus of the Zariski-closure of $\mathrm{Ad}_s(\Gamma)$. Then $G^S$ and $\Gamma$ satisfy the Mostow condition.*
(5) *Suppose that $G$ has a lattice $\Gamma$. Then $G$ is of (I)-type if and only if there exists a finite index subgroup $\tilde{\Gamma}$ of $\Gamma$ such that the restriction $\mathrm{Ad}_s|_{\tilde{\Gamma}}$ is trivial.*

If on $G$ instead of considering the product $\bullet_S$ we consider the product

$$a \tilde{\bullet}_S b = \pi_S(b)^{-1} ab$$

we can give an interpretation of the $S$-modification $\tilde{G}^S = (G, \tilde{\bullet}_S)$ in terms of crossed homomorphisms. Note that $\tilde{G}^S = (G, \tilde{\bullet}_S)$ has Lie algebra which is isomorphic to the Lie algebra of right-invariant vector fields on $G^S = (G, \bullet_S)$.

We recall the following

**Definition 3.3.** [25, (1.11)] A function $\phi : G_1 \to G_2$ between two groups is called a *crossed homomorphism* if there exists some function $\mu : G_1 \to \mathrm{Aut}(G_2)$ such that

$$\phi(gh) = \mu(h)(\phi(g))\phi(h)\,, \qquad \forall g, h \in G_1\,.$$

Note that the definition of the product $\tilde{\bullet}_S$ means that the identity $\mathrm{id}_G$ is a crossed isomorphism between $G$ and $G^S$, with $\mu : g \mapsto \pi_S(g)^{-1}$.

By [18, Theorem 5] if $\Gamma_1$ and $\Gamma_2$ are lattices in simply connected completely solvable Lie groups $G_1$ and $G_2$ respectively, then any isomorphism from $\Gamma_1$ and $\Gamma_2$ extends to an isomorphism from $G_1$ to $G_2$. This result does not generalize to the class of all solvable Lie groups, even for lattices $\Gamma_i$ such that $G_i$ and $\Gamma_i$ satisfy the Mostow condition. In [25] Witte proved an extension in terms of the so-called doubly-crossed homomorphisms, showing the following rigidity property:

*If $\Gamma_1$ and $\Gamma_2$ are lattices in simply connected solvable Lie groups $G_1$ and $G_2$, then any isomorphism from $\Gamma_1$ and $\Gamma_2$ extends to a $\Gamma_1$-equivariant doubly-crossed isomorphism from $G_1$ to $G_2$* (see [25, Corollary 7.4']).

In general, one can state the following

**Definition 3.4.** [25, Definition 7.3'] A map $\psi : G_1 \to G_2$ between two simply connected solvable Lie groups $G_1$ and $G_2$ is a *doubly-crossed homomorphism* if there are

(1) a solvable Lie group $G$;
(2) a crossed isomorphism $\phi_1 : G \to G_1$;
(3) a crossed isomorphism $\phi_2 : G \to G_2$

such that $\psi = \phi_2^{-1}\phi_1$, i.e. $\phi_2(g) = \psi(\phi_1(g))$, for every $g \in G$. For a subgroup $\Gamma_1$ of $G_1$, one says that a doubly-crossed homomorphism $\psi : G_1 \to G_2$ is $\Gamma_1$-equivariant if

$$\psi(g\gamma) = \psi(g)\psi(\gamma),$$

for every $g \in G_1$ and $\gamma \in \Gamma_1$.

To prove the rigidity property for the lattices Witte considered the following more restrictive definition:

*A map $\phi : G_1 \to G_2$ is a doubly-crossed homomorphism if there exist*

(1) *compact torus $T_i \subset \mathcal{A}(\mathrm{Ad}_{G_i}(G_i))$, $i = 1, 2$;*
(2) *a continuous homomorphism $\sigma_1 : G_1 \to T_1$ which gives a crossed homomorphism*

$$\Delta_1 : G_1 \to \Delta_1(G_1) \subset T_1 \ltimes G_1, g \longmapsto (\sigma_1(g), g);$$

(3) *a continuous homomorphism* $\sigma_2 \times \phi_2 : \Delta_1(G_1) \to T_2 \ltimes G_2$*, with* $\sigma_2(h) \in T_2$*, for every* $h \in G_1$*.*

*such that* $\phi = \Delta_1 \phi_2$*.*

As remarked by Witte, $\phi_2$ is a crossed homomorphism from $\Delta_1(G_1)$ to $G_2$ and $\Delta_1$ is a crossed isomorphism from $G_1$ to $\Delta_1(G_1)$, whose inverse is also a crossed isomorphism.

Thus, with respect to Definition 3.4, one is considering $G = \Delta_1(G)$ and $\phi_1 = \Delta_1^{-1} : \Delta_1(G_1) \to G_1$.

More precisely, Witte constructed the doubled crossed homomorphism between the two solvable Lie groups $G_1$ and $G_2$ in the following way. Let $T_2$ be a maximal compact torus of $\mathcal{A}(\mathrm{Ad}_{G_2} G_2)$, and let $T_2 = S_2 \times S_2^{\perp}$ be a direct-product decomposition of $T_2$. Consider the corresponding nilshadow $G_2' = \Delta_2(G_2)$ of $G_2$, and let $\sigma_2' : G_2 \to S_2^{\perp}$ be a homomorphism, with corresponding diagonal embedding $\Delta_2'(G_2')$. In general $\Delta_2'(G_2')$ may not be a diagonal embedding of $G_2$, since $\sigma_2'$ may not be trivial on $[G_2, G_2]$, though it is trivial on the subgroup $[G_2', G_2']$. Any isomorphism from a nilshadow $\Delta_1(G_1)$ to $\Delta_2'(G_2)$ induces a doubly-crossed isomorphism from $G_1$ to $G_2$.

Suppose that two compact solvmanifolds $G_1/\Gamma_1$ and $G_2/\Gamma_2$ are diffeomorphic. Then applying Witte's result an isomorphism $\alpha$ between $\Gamma_1$ and $\Gamma_2$ induces a $\Gamma_1$-equivariant doubly-crossed isomorphism from $G_1$ to $G_2$

$$\phi : G_1 \to G_2.$$

We can see the doubly-crossed isomorphism $\phi$ as a crossed homomorphism from a modification of the Lie groups $G_1$ and $G_2$. Indeed, by definition of doubly-crossed homomorphism we have

$$\phi(gh) = \sigma_2(h)\phi(\sigma_1(h)^{-1}(g))\phi(h),$$

for every $g, h \in G_1$, with $T_i$ compact torus in $\mathcal{A}(\mathrm{Ad}_{G_i}(G_i))$, and $\sigma_i : G_1 \to T_i$, $i = 1, 2$.

Note that using $\sigma_1$ one may consider the $T_1$-modification $\tilde{G}_1^{T_1}$ of $G_1$ with product

$$g \tilde{\bullet}_{T_1} h = \sigma_1(h) g h.$$

Since

$$\phi(\sigma_1(h)gh) = \sigma_2(h)\phi(g)\phi(h),$$

for every $g, h \in G_1$, the $\Gamma_1$-equivariant doubly-crossed isomorphism $\phi$ from $G_1$ to $G_2$ is then a crossed isomorphism from the modification $\tilde{G}_1^{T_1} = (G_1, \tilde{\bullet}_{T_1})$ to $G_2$ with

$$\mu = \sigma_2 : (G_1, \tilde{\bullet}_{T_1}) \to T_2 \subset \mathrm{Aut}(G_2).$$

Since $\Gamma_1 = \sigma_1(\Gamma_1)$, $\Gamma_1$ is also a lattice in $\tilde{G}_1^{T_1}$ and $\mu$ is $\Gamma_1$-invariant. By construction $\alpha$ extends to a homomorphism $\beta : \Delta_1(G_1) \to T_2 \ltimes G_2$ and thus to a homomorphism $\beta$ from $\tilde{G}_1^{T_1} = (G_1, \tilde{\bullet}_{T_1})$ to $T_2 \ltimes G_2$. This homomorphism is the extension of $\alpha : \Gamma_1 \to \Gamma_2$.

3.2. **Modification and invariant cohomology.** Let $\mathfrak{g}$ be the Lie algebra of $G$ and $\mathfrak{g}^S$ the Lie algebra of the $S$-modification $G^S$ with product $\bullet_S$ defined in (1).

Denote by $L_g^G$ the left translation of $g \in G$ on $G$ and by $L_g^{G^S}$ the left translation of $g$ in $G^S = (G, \bullet_S)$. Then $L_g^{G^S} = L_g^G \circ \pi_S(g)^{-1}$, for every $g \in G$. Hence

$$\mathfrak{g}^S = \{\tilde{X} \in C^\infty(TG) | X \in T_e G, \ (\tilde{X})_g = L_g^G \circ \pi_S(g)^{-1} X\}.$$

Since $\mathrm{Ad}_s : G \to \mathrm{Aut}(\mathfrak{g})$ is $\mathbb{C}$-diagonalizable, there exists a basis $X_1, \ldots, X_n$ of $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$ such that $\mathrm{Ad}_s = \mathrm{diag}\,(\alpha_1, \ldots, \alpha_n)$ for some characters $\alpha_1, \ldots, \alpha_n$. We also have $\pi_S = \mathrm{diag}\,(\beta_1, \ldots, \beta_n)$ for some characters $\beta_1, \ldots, \beta_n$. Then, since

$$\mathfrak{g}^S = \{\tilde{X} \in C^\infty(TG) | X \in T_e G, \ (\tilde{X})_g = L_g^G \circ \pi_S(g)^{-1} X\},$$

we obtain that $\beta_1^{-1} X_1, \ldots, \beta_n^{-1} X_n$ is a basis of $\mathfrak{g}^S \otimes \mathbb{C} = \mathfrak{g}^S_{\mathbb{C}}$.

Let $X_1, \ldots, X_n$ be a basis of $\mathfrak{g}_{\mathbb{C}}$ such that

$$\mathrm{Ad}_s = \mathrm{diag}\,(\alpha_1, \ldots, \alpha_n), \quad \pi_S = \mathrm{diag}\,(\beta_1, \ldots, \beta_n).$$

Then

$$\mathfrak{g}_{\mathbb{C}}^S = \mathrm{span} < \beta_1^{-1} X_1, \ldots, \beta_n^{-1} X_n >$$

and

$$\bigwedge^* \mathfrak{g}_{\mathbb{C}}^{S^*} = \bigwedge^* < \beta_1 x_1, \ldots, \beta_n x_n >,$$

where $x_1, \ldots, x_n$ is the dual basis of $X_1, \ldots, X_n$.

If $S = T$ we get

$$\bigwedge^* \mathfrak{u}_{\mathbb{C}}^* = \bigwedge^* < \alpha_1 x_1, \ldots, \alpha_n x_n >,$$

where $\mathfrak{u}$ is the Lie algebra of the nilshadow of $G$.

Let

$$A_\Gamma^* = A^*(G/\Gamma) \cap \bigwedge^* \mathfrak{u}_{\mathbb{C}}^*,$$

where $A^k(G/\Gamma)$ denotes the space of $k$-forms on $G/\Gamma$. Then

$$A_\Gamma^* = \mathrm{span} < \alpha_{i_1 \ldots i_p} x_{i_1} \wedge \ldots \wedge x_{i_p} \ \mid \ \alpha_{i_1 \ldots i_p}|_\Gamma = 1 >$$

In [16], it was proved that the inclusion $A_\Gamma^* \subset A^*(G/\Gamma)$ induces a cohomology isomorphism. Hence we have:

**Theorem 3.5.** *Let $G/\Gamma$ be a compact solvmanifold. If there exists a subtorus $S$ of $T$ such that $\pi_s(\Gamma) = 1$ and $A_\Gamma^* \subset \bigwedge (g^S)_{\mathbb{C}}^*$, then $H_{dR}^*(G/\Gamma) \cong H^*(\mathfrak{g}^S)$.*

**Remark 3.6.** Consider a finite index subgroup $\tilde{\Gamma} \subset \Gamma$ such that $S^\perp = \mathcal{A}(\mathrm{Ad}_s(\tilde{\Gamma}))$ (and hence $\mathcal{A}(\mathrm{Ad}(\tilde{\Gamma})))$ is connected. We take a sub-torus $S \subset T$ such that $T = S \times S^\perp$. Then we have $A_\Gamma^* \subset \bigwedge^* (\mathfrak{g}^S)_{\mathbb{C}}^*$. Hence Theorem 3.5 is a generalization of Theorem 2.5.

Indeed, consider the projections $p_S : T \to S$ and $p_{S^\perp} : T \to S^\perp$. Observe that $\pi_S = p_S \circ \mathrm{Ad}_s$. Let $D_n(\mathbb{C})$ denote the complex diagonal matrices. For

$$t_i \in \mathrm{Char}(D_n(\mathbb{C})) = \{t_1^{m_1} \cdots t_n^{m_n} | (t_1, \ldots, t_n) \in D_n(\mathbb{C})\},$$

we define $f_i = (t_i)_{|T}$, $g_i = (t_i)_{|S} \circ p_S$ and $h_i = (t_i)_{|S^\perp} \circ p_{S^\perp}$. We have $f_i \circ \mathrm{Ad}_s = \beta_i$, $g_i \circ \mathrm{Ad}_s = \beta_i$ and $f_i = g_i h_i$. Suppose $(\alpha_I)\lfloor_\Gamma = 1$ for some $I \subseteq \{1, \ldots, n\}$. Consider

$$f_I = g_I h_I.$$

Then, since $\mathrm{Ad}_s(\Gamma) \subset S^\perp$, we get $(g_I)_{|Ad_s(\tilde{\Gamma})} = 1$. Hence

$$1 = (\alpha_I)\lfloor_{\tilde{\Gamma}} = (f_I)_{\mathrm{Ad}_s(\tilde{\Gamma})} = (h_I)_{Ad_s(\tilde{\Gamma})}.$$

Since $S^\perp$ is Zariski-closure of $\mathrm{Ad}_s(\tilde{\Gamma})$, we obtain $h_I = 1$ and thus $f_I = g_I$. Hence if $(\alpha_I)\lfloor_\Gamma = 1$ for $I \subseteq \{1, \ldots, n\}$, then we have

$$\alpha_I = \beta_I.$$

This implies $A_\Gamma^* \subset \bigwedge (\mathfrak{g}^S)_{\mathbb{C}}^*$ and gives a proof of the above remark.

**Example 3.7.** Let $G = \mathbb{R} \ltimes_\phi \mathbb{R}^2$ such that $\phi(t) = \begin{pmatrix} \cos \pi t & -\sin \pi t \\ \sin \pi t & \cos \pi t \end{pmatrix}$. Then $G$ has two lattices $\Gamma_1 = \mathbb{Z} \ltimes \mathbb{Z}^2$ and $\Gamma_2 = 2\mathbb{Z} \ltimes \mathbb{Z}^2$. In the case of $\Gamma_1$, we take $S$ trivial. Then $A_\Gamma^* \subset \bigwedge (\mathfrak{g}^S)_{\mathbb{C}}^*$ and hence $H^*(G/\Gamma_1) \cong H^*(\mathfrak{g}^S) \cong H^*(\mathfrak{g})$. In the case of $\Gamma_2$, we take $S = T = Ad_s$. Then, $A_\Gamma^* \subset \bigwedge (\mathfrak{g}^S)_{\mathbb{C}}^*$ and hence $H^*(G/\Gamma_1) \cong H^*(\mathfrak{g}^S) \cong H^*(\mathbb{R}^3)$.

3.3. **Modification and invariant complex structures.** Let $J$ be a left-invariant complex structure on $G$ such that $J \circ \mathrm{Ad}_{sg} = \mathrm{Ad}_{sg} \circ J$ for every $g \in G$. Since $T$ is the Zariski closure of $\mathrm{Ad}_s(G)$, $t$ and $J$ commute, for every $t \in T$. Moreover, for any $\tilde{X} \in \mathfrak{g}^S$ we have

$$J_g(\tilde{X})_g = J_g \circ L_g^G \circ \pi_S(g)^{-1} X = L_g^G \circ \pi_S(g)^{-1}(J_e X) = L_g^{G^S}(J_e(\tilde{X})_e).$$

As a consequence

**Proposition 3.8.** *Let $J$ be a left-invariant complex structure on $G$ such that $J \circ \mathrm{Ad}_{sg} = \mathrm{Ad}_{sg} \circ J$ for every $g \in G$. Then $J$ can be considered as a left-invariant complex structure on the $S$-modification $G^S = (G, \bullet_S)$. Moreover for a lattice $\Gamma$ of $G$ such that the restriction $\pi_S|_\Gamma$ is trivial, the complex solvmanifold $(G/\Gamma, J)$ is biholomorphic to $G^S/\Gamma$ endowed with the invariant complex structure induced by the left-invariant complex structure $J$ on $G^S$.*

We can apply Proposition 3.8 to the following example.

**Example 3.9.** (**Nakamura manifold**) Consider the simply connected complex solvable Lie group $G$ defined by

$$G = \left\{ \begin{pmatrix} e^z & 0 & 0 & w_1 \\ 0 & e^{-z} & 0 & w_2 \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix}, \, w_1, w_2, z \in \mathbb{C} \right\}.$$

The Lie group $G$ is the semi-direct product $\mathbb{C} \ltimes_\varphi \mathbb{C}^2$, where

$$\varphi(z) = \begin{pmatrix} e^z & 0 \\ 0 & e^{-z} \end{pmatrix}, \qquad \forall z \in \mathbb{C}.$$

A basis of complex left-invariant 1-forms is given by

$$\phi_1 = dz, \, \phi_2 = e^{-z} dw_1, \, \phi_3 = e^z dw_2$$

and in terms of the real basis of left-invariant 1-forms $(e^1, \ldots, e^6)$ defined by

$$\phi_1 = e^1 + \sqrt{-1} e^2, \, \phi_2 = e^3 + \sqrt{-1} e^4, \, \phi_3 = e^5 + \sqrt{-1} e^6,$$

we obtain the structure equations:

$$\begin{cases} de^j = 0, \quad j = 1, 2, \\ de^3 = -e^{13} + e^{24}, \\ de^4 = -e^{14} - e^{23}, \\ de^5 = e^{15} - e^{26}, \\ de^6 = e^{16} + e^{25}, \end{cases}$$

where we denote by $e^{ij}$ the wedge product $e^i \wedge e^j$.

Let $B \in SL(2, \mathbb{Z})$ be a unimodular matrix with distinct real eigenvalues: $\lambda, \frac{1}{\lambda}$. Consider $t_0 = \log \lambda$, i.e. $e^{t_0} = \lambda$. Then there exists a matrix $P \in GL(2, \mathbb{R})$ such that

$$PBP^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

Let

$$L_{1,2\pi} = \mathbb{Z}[t_0, 2\pi i] = \{t_0 k + 2\pi h i, h, k \in \mathbb{Z}\},$$
$$L_2 = \left\{ P \begin{pmatrix} \mu \\ \alpha \end{pmatrix}, \mu, \alpha \in \mathbb{Z}[i] \right\}.$$

Then, by [23] $\Gamma = L_{1,2\pi} \ltimes_\varphi L_2$ is a lattice of $G$.

Since $G$ has trivial center, we have that $\mathrm{Ad}_G(G) \cong G$ and thus $\mathrm{Ad}_G(G)$ is a semidirect product $\mathbb{R}^2 \ltimes \mathbb{R}^4$. Moreover, for the Zariski closures of $\mathrm{Ad}_G(G)$ and $\mathrm{Ad}_G(\Gamma)$ we obtain

$$\begin{aligned}\mathcal{A}(\mathrm{Ad}_G(G)) &= (\mathbb{R}^{\#} \times S^1) \ltimes \mathbb{R}^4,\\ \mathcal{A}(\mathrm{Ad}_G(\Gamma)) &= \mathbb{R}^{\#} \ltimes \mathbb{R}^4,\end{aligned}$$

where the split torus $\mathbb{R}^{\#}$ corresponds to the action of $e^{\frac{1}{2}(z+\overline{z})}$ and the compact torus $S^1$ to the one of $e^{\frac{1}{2}(z-\overline{z})}$.

Therefore in this case $\mathcal{A}(\mathrm{Ad}_G(G)) = S^1\mathcal{A}(\mathrm{Ad}_G(\Gamma))$ and $\mathcal{A}(\mathrm{Ad}_G(\Gamma))$ is connected. By applying Theorem 2.5 there exists a simply connected normal subgroup $\tilde{G} = \Delta(G)$ of $S^1 \ltimes G$. The new Lie group $\tilde{G}$ is obtained by killing the action of $e^{\frac{1}{2}(z-\overline{z})}$. Indeed, we get that

$$\tilde{G} \cong \left\{ \begin{pmatrix} e^{\frac{1}{2}(z+\overline{z})} & 0 & 0 & w_1 \\ 0 & e^{-\frac{1}{2}(z+\overline{z})} & 0 & w_2 \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix}, w_1, w_2, z \in \mathbb{C} \right\}.$$

Note that $\tilde{G}$ is the modification $G^S$ of $G$, where the algebraic subtorus $S$ of $T = \mathcal{A}(\mathrm{Ad}_s(G)) \cong \mathbb{R}^{\#} \times S^1$ corresponds to the action of $e^{\frac{1}{2}(z-\overline{z})}$.

The diffeomorphism between $G/\Gamma$ and $\tilde{G}/\Gamma$ was already shown in [23]. Then in this case one has the isomorphism $H^*_{dR}(G/\Gamma) \cong H^*(\tilde{\mathfrak{g}})$, where $\tilde{\mathfrak{g}}$ denotes the Lie algebra of $\tilde{G}$ and the de Rham cohomology of the Nakamura manifold $G/\Gamma$ is not isomorphic to $H^*(\mathfrak{g})$ (see also [9]). The Nakamura manifold $G/\Gamma$ is a complex parallelizable manifold and it is endowed with the bi-invariant complex structure $J$ with $(1,0)$-forms $(\phi_1, \phi_2, \phi_3)$, so in particular we have $J \circ \mathrm{Ad}_{sg} = \mathrm{Ad}_{sg} \circ J$, for every $g \in G$. Applying Proposition 3.8 we have that $J$ can be viewed as a left-invariant complex structure on the $S$-modification $G^S = \tilde{G}$ and since the restriction $\pi_S|_\Gamma$ is trivial, the Nakamura manifold $(G/\Gamma, J)$ is biholomorphic to $(G^S/\Gamma, J)$. This property was already shown in [23].

## 4. Modifications and Holomorphic Mostow fibrations

Let $G$ be a simply connected solvable Lie group with a lattice $\Gamma$ and $\mathfrak{g}$ be the Lie algebra of $G$. Let $N$ be the nilradical of $G$. It is known that $\Gamma \cap N$ is a lattice of $N$ and $\Gamma/\Gamma \cap N$ is a lattice of the abelian Lie group $G/N$ (see [21]). The compact solvmanifold $G/\Gamma$ is a fiber bundle

$$N/\Gamma \cap N = N\Gamma/\Gamma \longrightarrow G/\Gamma \longrightarrow G/N\Gamma = (G/N)/(\Gamma/\Gamma \cap N)$$

over a torus with a compact nilmanifold $N/\Gamma \cap N$ as fiber. Here the identification $N/\Gamma \cap N = N\Gamma/\Gamma$ is given by the correspondence

$$n'\Gamma \cap N \in N/\Gamma \cap N \mapsto n'\Gamma \in N\Gamma/\Gamma. \tag{2}$$

This fiber bundle is called the *Mostow bundle* of $G/\Gamma$. Its structure group is $N\Gamma/\Gamma_0$, where $\Gamma_0$ is the largest normal subgroup of $\Gamma$ which is normal in $N\Gamma$, and the action is by left translations (see [22]).

For a maximal torus $T$ of $\mathcal{A}(\mathrm{Ad}(G))$, consider the splitting $T \ltimes G$. Take the centralizer $C = C_T(G)$. Then $C$ is a simply connected nilpotent subgroup of $G$ such that $G = C \cdot N$ (see [10]). Note that $N\Gamma = N \cdot (\Gamma \cap C)$.

Suppose that $N$ admits a left-invariant complex structure $J_N$. For $nc \in N \cdot (\Gamma \cap C)$, by using the correspondence (2), we have

$$nc \cdot (n'\Gamma \cap N) = ncn'c^{-1}\Gamma \cap N.$$

Hence the structure group $N\Gamma/\Gamma_0$ consists of holomorphic transformations if and only if for any $c \in \Gamma \cap C$ we have $\mathrm{Ad}_c \circ J_N = J_N \circ \mathrm{Ad}_c$ on $\mathfrak{n}$.

If $G$ has a left-invariant complex structure $J$ such that $J(\mathfrak{n}) = \mathfrak{n}$ and $\mathrm{Ad}_c \circ J_N = J_N \circ \mathrm{Ad}_c$ on $\mathfrak{n}$, for any $c \in C$, then the Mostow fibration of $G/\Gamma$ is holomorphic.

**Proposition 4.1.** *Suppose that for any $c \in \Gamma \cap C$ we have $\mathrm{Ad}_c \circ J_N = J_N \circ \mathrm{Ad}_c$ on $\mathfrak{n}$. Then there exist a finite index subgroup $\tilde{\Gamma} \subset \Gamma$, an $S$-modification $G^S$ containing $\tilde{\Gamma}$, where $S$ is a complement of a maximal torus of $\mathcal{A}(\mathrm{Ad}(\tilde{\Gamma}))$, and a simply connected nilpotent subgroup $C' \subset G^S$ such that for any $c' \in C'$ $\mathrm{Ad}_{c'} \circ J_N = J_N \circ \mathrm{Ad}_{c'}$ on $\mathfrak{n}$.*

*Proof.* Suppose that for any $c \in \Gamma \cap C$ we have $\mathrm{Ad}_c \circ J_N = J_N \circ \mathrm{Ad}_c$ on $\mathfrak{n}$. We can assume that $\mathcal{A}(\mathrm{Ad}(\Gamma))$ is connected, otherwise we can pass to finite index subgroup of $\Gamma$. Let $T$ be a maximal torus of $\mathcal{A}(\mathrm{Ad}(G))$ and $S^{\perp}$ a maximal torus of $\mathcal{A}(\mathrm{Ad}(\Gamma))$. Consider the semi-direct product $T \ltimes G$. Then this group is a real algebraic group as $T \ltimes G = T \ltimes U_G$, where $U_G$ is the nilshadow of $G$. By this, the nilradical $N$ of $G$ is a unipotent algebraic subgroup of $U_G$. Take the Zariski-closures $\mathcal{A}(C)$ and $\mathcal{A}(\Gamma \cap C)$ in $T \ltimes U_G$. Since for any $c \in \Gamma \cap C$ we have $\mathrm{Ad}_c \circ J_N = J_N \circ \mathrm{Ad}_c$ on $\mathfrak{n}$ and the action of $T \ltimes U_G$ on $\mathfrak{n}$ is algebraic, for any $c' \in \mathcal{A}(\Gamma \cap C)$ we have $\mathrm{Ad}_{c'} \circ J_N = J_N \circ \mathrm{Ad}_{c'}$. For the splitting $T \ltimes G = T \ltimes U_G$, we have $C = C_{T \ltimes U_G}(T) \cap G$ (see [10]). Hence we have $C \cap \Gamma = C_{T \ltimes U_G}(T) \cap \Gamma$. Since the Zariski closures of $\Gamma$ and $G$ in $T \ltimes U_G$ are $S^{\perp} \ltimes U_G$ and $T \ltimes U_G$ respectively, $\mathcal{A}(C)$ and $\mathcal{A}(\Gamma \cap C)$ have the same unipotent radical $U' = U_G \cap C_{T \ltimes U_G}(T)$ and we have $\mathcal{A}(C) = T \ltimes U'$ and $\mathcal{A}(\Gamma \cap C) = S^{\perp} \ltimes U'$. Consider the modification $G^S$. Then $G^S = C' \cdot N$ with $C' = \{\pi_S(c)^{-1} \cdot c | c \in C\}$. Since $C' \subset S^{\perp} \ltimes U' = \mathcal{A}(\Gamma \cap C)$, for every $c' \in C'$ we have $\mathrm{Ad}_{c'} \circ J_N = J_N \circ \mathrm{Ad}_{c'}$. □

As a consequence, we can state the following

**Corollary 4.2.** *Let $G/\Gamma$ be a compact solvmanifold. Suppose that $G$ has a left-invariant complex structure $J$ such that $J(\mathfrak{n}) = \mathfrak{n}$. Then the following two conditions are equivalent.*
(1) *For any $c \in C$ $\mathrm{Ad}_c \circ J_N = J_N \circ \mathrm{Ad}_c$ on $\mathfrak{n}$.*
(2) *For any $g \in G$, $\mathrm{Ad}_{sg} \circ J_N = J_N \circ \mathrm{Ad}_{sg}$ and the Mostow fibration of $G/\Gamma$ is holomorphic.*

*Proof.* Since $\mathrm{Ad}_s$ is identified with the map $G = C \cdot N \ni cn \mapsto (\mathrm{Ad}_c)_s \in \mathrm{Aut}(\mathfrak{g})$, the assertion (1) ⇒ (2) also follows.

Suppose that the condition (2) holds. For $T = \mathcal{A}(\mathrm{Ad}_S(G))$, using the splitting $T \ltimes G = T \ltimes U$, we have $C = C_{T \ltimes U}(T) \cap G$. Like in the proof of Proposition 4.1, we have

$$\pi_S(c)^{-1} \circ \mathrm{Ad}_c \circ J_N = J_N \circ \pi_S(c)^{-1} \circ \mathrm{Ad}_c.$$

By $\mathrm{Ad}_{sg} \circ J = J \circ \mathrm{Ad}_{sg}$ and $\pi_S(c) \in T$ we have $\pi_S(c) \circ J_N = J_N \circ \pi_S(c)$. Hence we obtain

$$\mathrm{Ad}_c \circ J_N = \pi_S(c) \circ \pi_S(c)^{-1} \circ \mathrm{Ad}_c \circ J_N = J_N \circ \pi_S(c) \circ \pi_S(c)^{-1} \circ \mathrm{Ad}_c = J_N \circ \mathrm{Ad}_c \,,$$

and therefore the assertion (2) ⇒ (1) follows. □

Using Proposition 4.1 it is possible to obtain a left-invariant complex structure on a modification of $G$ starting with (a non necessarily integrable) almost complex structure on $G = \mathbb{C}^n \ltimes_\phi N$.

**Corollary 4.3.** *Let $G$ be a simply connected solvable Lie group with nilradical $N$. We suppose that*
(1) *$G = \mathbb{C}^n \ltimes_\phi N$ and $G$ admits a lattice $\Gamma = \Gamma_{\mathbb{C}^n} \ltimes \Gamma_N$.*
(2) *$N$ has a left-invariant complex structure $J_N$ such that the Mostow fibration*

$$N/\Gamma_N \to G/\Gamma \to \mathbb{C}^n/\Gamma_{\mathbb{C}^n}$$

*is holomorphic.*[1]

[1] However we do not assume that the almost complex structure $J_{\mathbb{C}^n} \oplus J_N$ on $\mathfrak{g} = \mathbb{C}^n \ltimes_\phi \mathfrak{n}$ is integrable.

*Then there exists a finite index subgroup $\tilde{\Gamma} \subset \Gamma$ and a $S$-modification $G^S = \mathbb{C}^n \ltimes_{\tilde{\phi}} N$ containing $\tilde{\Gamma}$ such that for any $t \in \mathbb{C}^n$ we have $\tilde{\phi}(t) \circ J_N = J_N \circ \tilde{\phi}(t)$. In particular the almost complex structure $J_{\mathbb{C}^n} \oplus J_N$ on $\mathfrak{g}^S = \mathbb{C}^n \ltimes_{\tilde{\phi}} \mathfrak{n}$ is integrable.*

**Example 4.4.** Let $G = \mathbb{R}^2 \ltimes_\phi \mathbb{R}^8$ with

$$\phi(x,y) = \begin{pmatrix} e^x\cos y & -e^x\sin y & 0 & 0 & 0 & 0 & 0 & 0 \\ e^x\sin y & e^x\cos y & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^x\cos y & e^x\sin y & 0 & 0 & 0 & 0 \\ 0 & 0 & -e^x\sin y & e^x\cos y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-x}\cos y & -e^{-x}\sin y & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-x}\sin y & e^{-x}\cos y & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{-x}\cos y & e^{-x}\sin y \\ 0 & 0 & 0 & 0 & 0 & 0 & -e^{-x}\sin y & e^{-x}\cos y \end{pmatrix}.$$

The Lie algebra $\mathfrak{g}$ of $G$ is then $\mathfrak{g} = \langle X, Y\rangle \ltimes_\phi \mathfrak{n}$ where $\mathfrak{n} = \langle V_1, W_1, V_2, W_2, V_3, W_3, V_4, W_4\rangle$ with

$$[X,V_1] = V_1,\ \ [X,W_1] = W_1,\ \ [Y,V_1] = W_1,\ \ [Y,W_1] = -V_1,$$
$$[X,V_2] = V_2,\ \ [X,W_2] = W_2,\ \ [Y,V_2] = -W_2,\ \ [Y,W_2] = V_2,$$
$$[X,V_3] = -V_3,\ \ [X,W_3] = -W_3,\ \ [Y,V_3] = W_3,\ \ [Y,W_3] = -V_3,$$
$$[X,V_4] = -V_4,\ \ [X,W_4] = -W_4,\ \ [Y,V_4] = -W_4,\ \ [Y,W_4] = V_4.$$

Take on $\mathbb{R}^8$ the left-invariant complex structure $J_{\mathbb{R}^8}$ given by

$$J_{\mathbb{R}^8}V_1 = V_2,\ \ J_{\mathbb{R}^8}W_1 = W_2,\ \ J_{\mathbb{R}^8}V_3 = V_4,\ \ J_{\mathbb{R}^8}W_3 = W_4.$$

Then, since $[Y, J_{\mathbb{R}^8}V_1] = -W_2$ and $J_{\mathbb{R}^8}[Y,V_1] = W_2$, for $y \neq 0$ we have $\phi(0,y)\circ J_{\mathbb{R}^8} \neq J_{\mathbb{R}^8}\circ\phi(0,y)$ and for $J_{\mathbb{R}^2}$ with $J_{\mathbb{R}^2}(X) = Y$ the almost complex structure $J_{\mathbb{R}^2} \oplus J_{\mathbb{R}^8}$ is not integrable.

We can take a lattice $\Gamma = (a\mathbb{Z} + 2\pi\sqrt{-1}\mathbb{Z}) \ltimes \Gamma''$ for some $a \in \mathbb{R}$ and some lattice $\Gamma''$ in $\mathbb{R}^8$. Then we get a modification $G^S = \mathbb{R}^2 \ltimes_{\tilde{\phi}} \mathbb{R}^8$ with

$$\tilde{\phi}(x,y) = \begin{pmatrix} e^x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^x & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-x} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-x} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{-x} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-x} \end{pmatrix}.$$

Then $G^S$ contains $\Gamma$ and we have $\tilde{\phi}(x,y)\circ J_{\mathbb{R}^8} = J_{\mathbb{R}^8}\circ\tilde{\phi}(x,y)$ for every $(x,y) \in \mathbb{R}^2$.

## 5. Dolbeault cohomology

The aim of this section is to construct a differential bigraded algebra (DBA) which allows to compute the Dolbeault cohomology of a compact solvmanifold $(G/\Gamma, J)$ endowed with an invariant complex structure $J$, in the case that the Mostow fibration is holomorphic and $J$ commutes with $\mathrm{Ad}_s$.

We first recall some results in [15]. Consider a solvable Lie group $G$ which is a semi-direct product of the form $\mathbb{C}^n \ltimes_\phi N$ where:

- $N$ is a simply connected nilpotent Lie group with the Lie algebra $\mathfrak{n}$ and a left-invariant complex structure $J_N$;
- for any $t \in \mathbb{C}^n$, $\phi(t)$ is a holomorphic automorphism of $(N, J_N)$;
- $\phi$ induces a semi-simple action on the Lie algebra $\mathfrak{n}$ of $N$.

- $G$ has a lattice $\Gamma$ (then $\Gamma$ can be written by $\Gamma=\Gamma'\ltimes_\phi\Gamma''$ such that $\Gamma'$ and $\Gamma''$ are lattices of $\mathbb{C}^n$ and $N$ respectively and for any $t\in\Gamma'$ the action $\phi(t)$ preserves $\Gamma''$);
- the inclusion $\bigwedge^{*,*}\mathfrak{n}^*\subset A^{*,*}(N/\Gamma'')$ induces an isomorphism
$$H^{*,*}_{\bar\partial}(\mathfrak{n})\cong H^{*,*}_{\bar\partial}(N/\Gamma'')\,.$$

Then in [15] by using the Borel spectral sequence (see [12]) of the holomorphic fibration
$$N/\Gamma_N\to G/\Gamma\to\mathbb{C}^n/\Gamma_{\mathbb{C}^n}$$
determined by the splitting (in fact this spectral sequence is degenerate at $E_2$ (see [15, Section. 4]), it was constructed an explicit finite dimensional sub-DBA of the Dolbeault complex $A^{*,*}(G/\Gamma)$ which computes the Dolbeault cohomology of $(G/\Gamma,J)$.

In this section, in order to compute the Dolbeault cohomology of complex solvmanifolds of non-splitting type, we consider the condition that the Mostow fibration is holomorphic.

Given a complex representation $V_\rho$ of a simply connected complex abelian Lie group we define as in [21] the holomorphic flat bundle $E_\rho=(A\times V_\rho)/\Gamma$ given by the equivalence relation $(\gamma g,\rho(\gamma)v)\sim(g,v)$ for $g\in A$, $v\in V_\rho$ and $\gamma\in\Gamma$. We can prove the following

**Lemma 5.1.** *Let $A$ be a simply connected complex abelian Lie group with a lattice $\Gamma$, $\mathfrak{a}$ the Lie algebra of $A$ and $\rho:A\to GL(V_\rho)$ a representation of $A$ on a complex vector space $V_\rho$. Consider the generalized weight decomposition (i.e. the weight decomposition of the semisimple part $\rho_s$ of $\rho$)*
$$V_\rho=\bigoplus V_{\rho_{\alpha_i}}\,.$$
*Take unitary characters $\beta_i$ such that $\alpha_i\beta_i^{-1}$ are holomorphic on $A$ as in* [15, Lemma 2.2].

*Then we have an isomorphism*
$$H^{*,*}(A/\Gamma,E_\rho)\cong\bigoplus_{\beta_i|\Gamma=1}H^{*,*}(\mathfrak{a},V_{\beta_i^{-1}\rho_{\alpha_i}})\,,$$
*where $E_\rho$ is the holomorphic flat vector bundle induced by the representation $\rho$.*

*Proof.* Let $E_{\rho_{\alpha_i}}$ be the holomorphic flat vector bundle induced by the representation $\rho_{\alpha_i}$. Since $\rho_{\alpha_i}$ is triangularizable, we have a sequence
$$E_{\rho_{\alpha_i}}=E^0_{\rho_{\alpha_i}}\supset E^1_{\rho_{\alpha_i}}\supset E^2_{\rho_{\alpha_i}}\cdots\supset E^j_{\rho_{\alpha_i}}=(A/\Gamma)\times\{0\}$$
of sub-bundles such that $E^k_{\rho_{\alpha_i}}/E^{k+1}_{\rho_{\alpha_i}}\cong L_{\alpha_i}$ where $L_{\alpha_i}$ is the holomorphic flat line bundle defined by the character $\alpha_i$. It is known that
$$H^{*,*}(A/\Gamma,L_{\alpha_i})=0$$
if $\beta_i|_\Gamma\neq1$ see [15, Lemma 2.3, Proposition 2.4]. Hence inductively if $\beta_i|_\Gamma\neq1$, we get
$$H^{*,*}(A/\Gamma,E_{\rho_{\alpha_i}})=0.$$
If $\beta_i|_\Gamma=1$, we have
$$H^{*,*}(A/\Gamma,E_{\rho_{\alpha_i}})\cong H^{*,*}(A/\Gamma,E_{\beta_i^{-1}\rho_{\alpha_i}}).$$
Consider the natural inclusion
$$\bigwedge^{*,*}\mathfrak{a}^*\otimes V_{\beta_i^{-1}\rho_{\alpha_i}}\subset A^{*,*}(A/\Gamma,E_{\beta_i^{-1}\rho_{\alpha_i}}).$$
Then since the line bundle $L_{\beta_i^{-1}\alpha_i}$ is trivial, inductively this inclusion induces a cohomology isomorphism. Hence the lemma follows. $\square$

Next we apply the previous Lemma to a compact complex solvmanifold $(G/\Gamma,J)$, when the Mostow fibration of $G/\Gamma$ is holomorphic.

**Theorem 5.2.** *Let $G$ be a simply connected solvable Lie group with a left-invariant complex structure $J$, a lattice $\Gamma$ and the nilradical $N$. Let $\mathfrak{g}$ be the Lie algebra of $G$ and $\mathfrak{n}$ be the Lie algebra of $N$. Suppose $J\mathfrak{n}\subseteq\mathfrak{n}$ and hence $J$ induces invariant complex structures on the nilmanifold $N/\Gamma\cap N$ and on the torus $G/N\Gamma$. Assume furthermore that $\mathrm{Ad}_{sg}\circ J=J\circ\mathrm{Ad}_{sg}$ for any $g\in G$ and that the Mostow fibration*

$$N/\Gamma\cap N\to G/\Gamma\to G/N\Gamma$$

*is holomorphic. We suppose that the inclusion $\bigwedge^{*,*}\mathfrak{n}^*\subset A^{*,*}(N/\Gamma\cap N)$ induces an isomorphism $H^{*,*}_{\bar\partial}(N/\Gamma\cap N)\cong H^{*,*}_{\bar\partial}(\mathfrak{n})$.*

*Under these assumptions, take a basis $X_1,\dots,X_n,Y_1,\dots,Y_m$ of $\mathfrak{g}^{1,0}$ such that $Y_1,\dots,Y_m$ is a basis of $\mathfrak{n}^{1,0}$ and $\mathrm{Ad}_{sg}Y_i=\alpha_i(g)Y_i$ for any $g\in G$. Take unitary charcters $\beta_i$ and $\gamma_i$ such that $\alpha_i\beta_i^{-1}$ and $\bar\alpha_i\gamma^{-1}$ are holomorphic on $G/N$ as in* [15, Lemma 2.2]. *Let $x_1,\dots,x_n,y_1\dots,y_m$ be the dual basis of $\mathfrak{g}^{1,0}$. Define the differential bigraded algebra (DBA)*

$$B^{p,q}_{\Gamma}=\bigoplus_{a+c=p,b+d=q}\bigwedge^{a}\langle x_1,\dots,x_n\rangle\otimes\bigwedge^{b}\langle\bar x_1,\dots,\bar x_n\rangle\otimes\left\langle\beta_I\gamma_Jy_I\wedge\bar y_J\,\middle|\,\begin{array}{c}|I|=c,\,|J|=d\\(\beta_J\gamma_L)_{|\Gamma}=1\end{array}\right\rangle$$

*Then the inclusion $B^{*,*}_{\Gamma}\subset A^{*,*}(G/\Gamma)$ induces a cohomology isomorphism.*

*Proof.* By

$$A^{*,*}(G/\Gamma)=C^\infty(G/\Gamma)\otimes\bigwedge^{*,*}\mathfrak{g}^*$$

and

$$\bigwedge\langle x_1,\dots,x_n,y_1,\dots,y_m\rangle\otimes\bigwedge\langle\bar x_1,\dots,\bar x_n,\bar y_1,\dots,\bar y_m\rangle=\bigwedge^{*,*}\mathfrak{g}^*,$$

we define the filtration

$$F^rA^{p,q}(G/\Gamma)=\bigoplus_{\substack{a+c=p,\\b+d=q,\\a+b\ge r}}C^\infty(G/\Gamma)\otimes\bigwedge^{a}\langle x_1,\dots,x_n\rangle\otimes\bigwedge^{b}\langle\bar x_1,\dots,\bar x_n\rangle\otimes\bigwedge^{c}\langle y_1,\dots,y_m\rangle\otimes\bigwedge^{d}\langle\bar y_1,\dots,\bar y_m\rangle.$$

This filtration yields the Borel spectral sequence ${}^*E^{*,*}_*$ of the holomorphic Mostow fibration (see [12]). We have

$${}^pE^{p,q}_1=\bigoplus_i A^{i,s-i}(G/\Gamma,\mathbf{H}^{p-i,t-i}(N/N\cap\Gamma))$$

and

$${}^pE^{p,q}_2=\bigoplus_i H^{i,s-i}(G/\Gamma,\mathbf{H}^{p-i,t-i}(N/N\cap\Gamma)).$$

By the filtration, associated to $B^{p,q}_\Gamma$ we have

$$F^rB^{p,q}_{\Gamma}=\bigoplus_{\substack{a+c=p,\\b+d=q,\\a+b\ge r}}\bigwedge^{a}\langle x_1,\dots,x_n\rangle\otimes\bigwedge^{b}\langle\bar x_1,\dots,\bar x_n\rangle\otimes\left\langle\beta_I\gamma_Jy_I\wedge\bar y_J\,\middle|\,\begin{array}{c}|I|=c,\,|J|=d\\(\beta_J\gamma_L)_{|\Gamma}=1\end{array}\right\rangle$$

and $\bar\partial F^rB^{p,q}_\Gamma\subset F^rB^{p,q+1}_\Gamma$. Hence the filtration induces the spectral sequence ${}^*\tilde E^{*,*}_*$ of $B^{*,*}_\Gamma$ and the homomorphism

$$\hat i:\tilde E^{*,*}_*\to E^{*,*}_*$$

determined by the inclusion $i:B^{*,*}_\Gamma\to A^{*,*}(G/\Gamma)$. We will prove that this homomorphism is an isomorphism at $E_2$-term.

We first prove that this homomorphism is injective. Since $G$ is unimodular, we can take a bi-invariant Haar measure $\eta$ such that $\int_{G/\Gamma}\eta=1$. Consider the map $\phi: A^{*,*}(G/\Gamma)\to B^{*,*}_{\Gamma}$ defined by

$$\phi\left(\sum f_{IJKL}x_I\wedge y_J\wedge \bar{x}_K\wedge \bar{y}_L\right)=\sum_{(\beta_I\gamma_J)_{|\Gamma}=1}\left(\int_{G/\Gamma}\frac{f_{IJKL}}{\beta_I\gamma_J}\eta\right)\beta_I\gamma_J x_I\wedge y_J\wedge \bar{x}_K\wedge \bar{y}_L$$

of the terms $\sum f_{IJKL}x_I\wedge y_J\wedge \bar{x}_K\wedge \bar{y}_L$. It is known that for any $\mathcal{C}^\infty$ function $F$ on $G/\Gamma$ and any left-invariant vector field $A$, we have $\int_{G/\Gamma}A(F)\eta=0$ (see [3]). Hence we get

$$\begin{aligned}\phi(\bar\partial(\textstyle\sum f_{IJKL}x_I\wedge y_J\wedge \bar{x}_K\wedge \bar{y}_L) &= \textstyle\sum_{(\beta_I\gamma_J)_{|\Gamma}=1}\left(\int_{G/\Gamma}\frac{f_{IJKL}}{\beta_I\gamma_J}\eta\right)\bar\partial(\beta_I\gamma_J x_I\wedge y_J\wedge \bar{x}_K\wedge \bar{y}_L)\\ &=\bar\partial(\phi(\textstyle\sum f_{IJKL}x_I\wedge y_J\wedge \bar{x}_K\wedge \bar{y}_L))\end{aligned}$$

and consequently $\phi$ is a homomorphism of cochain complexes. By the definition of $\phi$, we have $\phi(F^rA^{p,q}(G/\Gamma))\subset F^rB^{p,q}_{\Gamma}$ and $\phi\circ i=id_{B^{*,*}_{\Gamma}}$ for the inclusion $i: B^{*,*}_{\Gamma}\to A^{*,*}(G/\Gamma)$. Hence $\phi$ induces a homomorphism $\hat\phi: E^{*,*}_{*}\to \tilde{E}^{*,*}_{*}$ such that $\hat\phi\circ\hat\imath=id_{E^{*,*}_{*}}$ and for any term, $\hat\imath:\tilde{E}^{*,*}_{*}\to E^{*,*}_{*}$ is injective.

Consider the action $\rho: G/N\to \mathrm{Aut}(H^{*,*}(\mathfrak{n}))$ induced by the adjoint action of $G$ on $\mathfrak{n}$. Then since $H^{*,*}(N/N\cap\Gamma)\cong H^{*,*}(\mathfrak{n})$, we get that $H^{*,*}(N/N\cap\Gamma)$ is the holomorphic flat bundle induced by $\rho$. Take the generalized weight decomposition

$$H^{*,*}(\mathfrak{n})=\bigoplus V_{\rho_{\delta_i}}$$

and take the unitary characters $\epsilon_i$ such that $\delta_i\epsilon_i^{-1}$ are holomorphic. By Lemma 5.1, we have

$$\bigoplus {}^*E_2^{*,*}=\bigoplus H^{*,*}(G/N\Gamma,\mathbf{H}^{*,*}(N/N\cap\Gamma))\cong H^{*,*}\left(\mathfrak{g}/\mathfrak{n},\bigoplus_{\epsilon_i|_\Gamma=1}V_{\epsilon_i^{-1}\rho_{\delta_i}}\right).$$

Observe that $\rho=\mathrm{Ad}_s$ on $G/N\Gamma$. Let $\sigma: G/N\to\mathrm{Aut}(\bigwedge^{*,*}\mathfrak{n})$ be the action induced by $\mathrm{Ad}_s$ and

$$\bigwedge^{*,*}\mathfrak{n}=\bigoplus V_{\sigma_{\delta_i}}$$

the generalized weight decomposition. Then $V_{\sigma_{\delta_i}}$ are sub-cochain complexes of $\bigwedge^{*,*}\mathfrak{n}$ and this decomposition is a direct sum of cochain complexes. Then we have

$$B^{*,*}_{\Gamma}=\bigwedge^{*}\langle x_1,\dots,x_n\rangle\otimes\bigwedge^{*}\langle\bar{x}_1,\dots,\bar{x}_n\rangle\otimes\bigoplus_{\epsilon_i|_\Gamma=1}\epsilon_iV_{\sigma_{\delta_i}}.$$

Since $\sigma: G/N\to\mathrm{Aut}(\bigwedge^{*,*}\mathfrak{n})$ induces the semisimple part of the action $\rho: G/N\to\mathrm{Aut}(H^{*,*}(\mathfrak{n}))$, we have

$$H^{*,*}(V_{\sigma_{\delta_i}})\cong V_{\rho_{\delta_i}}\ ,$$

and consequently

$$\bigoplus {}^*\tilde{E}_1^{*,*}=\bigoplus\bigwedge^{*}\langle x_1,\dots,x_n\rangle\otimes\bigwedge^{*}\langle\bar{x}_1,\dots,\bar{x}_n\rangle\otimes\epsilon_iV_{\rho_{\delta_i}}\cong\bigoplus_{\epsilon_i|_\Gamma=1}\bigwedge^{*,*}(\mathfrak{g}/\mathfrak{n})\otimes V_{\epsilon_i^{-1}\rho_{\delta_i}}$$

and

$$\bigoplus {}^*\tilde{E}_2^{*,*}\cong H^{*,*}\left(\mathfrak{g}/\mathfrak{n},\bigoplus_{\epsilon_i|_\Gamma=1}V_{\epsilon_i^{-1}\rho_{\delta_i}}\right).$$

Hence the theorem follows. $\square$

**Example 5.3.** Let $\mathfrak{g} = \mathrm{span}\langle A_1, A_2, B_1, B_2, C_1, C_2, C_3, C_4, D_1, D_2, D_3, D_4\rangle$ such that

$$[A_1, A_2] = B_1,$$
$$[A_1, C_1] = k_0C_1,\ [A_1, C_2] = k_0C_2,\ [A_1, C_3] = -k_0C_3,\ [A_1, C_4] = -k_0C_4,$$
$$[A_2, D_1] = k_0D_1,\ [A_2, D_2] = k_0D_2,\ [A_2, D_3] = -k_0D_3,\ [A_1, D_4] = -k_0D_4,$$

where $k_0$ is a real constant number so that the matrix $\begin{pmatrix} e^{k_0} & 0 \\ 0 & e^{-k_0} \end{pmatrix}$ is conjugate to an element of $SL(2, \mathbb{Z})$. Then we have the nilradical $\mathfrak{n} = \langle B_1, B_2, C_1, C_2, C_3, C_4, D_1, D_2, D_3, D_4\rangle$ and the extension

$$0 \to \mathfrak{n} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{n} \to 0$$

cannot split. Consider the complex structure $J$ on $\mathfrak{g}$ defined by

$$JA_1 = A_2,\ JB_1 = B_2,\ JC_1 = C_2,\ JC_3 = C_4,\ JD_1 = D_2,\ JD_3 = D_4.$$

Let $G$ be the simply connected Lie group with the Lie algebra $\mathfrak{g}$ and $N$ the nilradical. Then $G$ can not split as $G = \mathbb{C} \ltimes N$, but on the other hand, we have

$$G = (H_3(\mathbb{R}) \times \mathbb{R}) \ltimes_\phi \mathbb{C}^4\,,$$

where

$$\phi\left(\begin{pmatrix} 1 & s_1 & t_1 \\ 0 & 1 & s_2 \\ 0 & 0 & 1 \end{pmatrix}, t_2\right) = \begin{pmatrix} e^{k_0 s_1} & 0 & 0 & 0 \\ 0 & e^{-k_0 s_1} & 0 & 0 \\ 0 & 0 & e^{k_0 s_2} & 0 \\ 0 & 0 & 0 & e^{-k_0 s_2} \end{pmatrix}$$

and $H_3(\mathbb{R})$ is the three dimensional Heisenberg group. $G$ has a lattice $\Gamma = (H_3(\mathbb{Z}) \times \mathbb{Z}) \ltimes_\phi \Gamma_{\mathbb{C}^4}$. We may take $V = \mathrm{span}\langle A_1,\ A_2\rangle$ such that $\mathfrak{g} = V \oplus \mathfrak{n}$ for which $\mathrm{ad}(A)_s(B) = 0$ for any $A, B \in V$. The nilpotent subgroup $C = H_3(\mathbb{R}) \times \mathbb{R}$ is a nilpotent complement of $N$ such that $G = C \cdot N$. Then by Corollary 4.2, $(G/\Gamma, J)$ satisfies the assumption in Theorem 5.2. Let $X_1 = A_1 - \sqrt{-1}A_2$, $Y_1 = B_1 - \sqrt{-1}B_2$, $Y_2 = C_1 - \sqrt{-1}C_2$, $Y_3 = C_3 - \sqrt{-1}C_4$, $Y_4 = D_1 - \sqrt{-1}D_2$ and $Y_5 = D_3 - \sqrt{-1}D_4$. Then we have $\mathfrak{g}^{1,0} = \mathrm{span}\langle X_1,\ Y_1,\ Y_2,\ Y_3,\ Y_4,\ Y_5\rangle$ and $\mathfrak{n}^{1,0} = \mathrm{span}\langle Y_1,\ Y_2,\ Y_3,\ Y_4,\ Y_5\rangle$. Take the dual basis $x_1, y_1, y_2, y_3, y_4, y_5$ of $(\mathfrak{g}^{1,0})^*$. We have

$$B^{*,*}_\Gamma = \bigwedge\langle x_1,\ \bar{x}_1\rangle \otimes \bigwedge\langle y_1,\ \bar{y}_1,\ y_2 \wedge y_3,\ y_2 \wedge \bar{y}_3,\ y_3 \wedge \bar{y}_2,\ \bar{y}_2 \wedge \bar{y}_3, y_4 \wedge y_5,\ y_4 \wedge \bar{y}_5,\ y_5 \wedge \bar{y}_4,\ \bar{y}_4 \wedge \bar{y}_5\rangle.$$

We have

$$H^{*,*}(G/\Gamma) \cong H^{*,*}((H_3(\mathbb{R})\times\mathbb{R})/_{(H_3(\mathbb{Z})\times\mathbb{Z})})\otimes\bigwedge\langle y_2\wedge y_3,\ y_2\wedge\bar{y}_3,\ y_3\wedge\bar{y}_2,\ \bar{y}_2\wedge\bar{y}_3, y_4\wedge y_5,\ y_4\wedge\bar{y}_5,\ y_5\wedge\bar{y}_4,\ \bar{y}_4\wedge\bar{y}_5\rangle.$$

**Remark 5.4.** Suppose $G$ is the semi-direct product $\mathbb{C}^n \ltimes_\phi N$ such that $N$ is a simply connected nilpotent Lie group with the Lie algebra $\mathfrak{n}$. Then in general $N$ is not the nilradical of $G$. On the other hand, in this case, we have $\mathrm{Ad}_s = \mathrm{id}_{\mathbb{C}^n} \oplus \phi_s$ on $\mathfrak{g} = \mathbb{C}^n \ltimes \mathfrak{n}$ where $\phi_s$ is a semi-simple part of $\phi : \mathbb{C}^n \to \mathrm{Aut}(\mathfrak{n})$. By a similar proof as the one of Theorem 5.2, as a generalization of the result in [15] we have the following theorem.

**Theorem 5.5.** *Consider a solvable Lie group $G$ which is the semi-direct product $\mathbb{C}^n \ltimes_\phi N$ where:*

- *$N$ is a simply connected nilpotent Lie group with the Lie algebra $\mathfrak{n}$ and a left-invariant complex structure $J_N$.*
- *For any $t \in \mathbb{C}^n$, $\phi(t)$ is a holomorphic automorphism of $(N, J_N)$.*[2]
- *$G$ has a lattice $\Gamma$. (Then $\Gamma$ can be written by $\Gamma = \Gamma' \ltimes_\phi \Gamma''$ such that $\Gamma'$ and $\Gamma''$ are lattices of $\mathbb{C}^n$ and $N$ respectively and for any $t \in \Gamma'$ the action $\phi(t)$ preserves $\Gamma''$).*

[2] Note we do not need to suppose $\phi$ is semi-simple.

- *The inclusion* $\bigwedge^{*,*}\mathfrak{n}^* \subset A^{*,*}(N/\Gamma'')$ *induces an isomorphism*
$$H^{*,*}_{\bar\partial}(\mathfrak{n}) \cong H^{*,*}_{\bar\partial}(N/\Gamma'').$$

*Take a basis* $X_1,\dots,X_n,Y_1,\dots,Y_m$ *of* $\mathfrak{g}^{1,0}$ *such that* $Y_1,\dots,Y_m$ *is a basis of* $\mathfrak{n}^{1,0}$ *and* $\phi_s(g)Y_i = \alpha_i(g)Y_i$ *for any* $g \in \mathbb{C}^n$. *Take unitary characters* $\beta_i$ *and* $\gamma_i$ *such that* $\alpha_i\beta_i^{-1}$ *and* $\bar\alpha_i\gamma^{-1}$ *are holomorphic on* $\mathbb{C}^n$ *as in* [15, Lemma 2.2]. *Let* $x_1,\dots,x_n,y_1\dots,y_m$ *be the dual basis of* $\mathfrak{g}^{1,0}$. *Define the differential bigraded algebra (DBA)*

$$B^{p,q}_\Gamma = \bigoplus_{a+c=p,b+d=q} \bigwedge^a\langle x_1,\dots,x_n\rangle \otimes \bigwedge^b\langle \bar x_1,\dots,\bar x_n\rangle \otimes \left\langle \beta_I\gamma_J y_I\wedge \bar y_J \middle| \begin{array}{c} |I|=c,\, |J|=d \\ (\beta_J\gamma_L)_{|\Gamma}=1 \end{array}\right\rangle$$

*Then the inclusion* $B^{*,*}_\Gamma \subset A^{*,*}(G/\Gamma)$ *induces a cohomology isomorphism.*

## 6. Dolbeault cohomology and modifications

Using the DBA described in the previous Section, we will show that for some cases the Dolbeault cohomology of $(G/\Gamma, J)$ is isomorphic to the invariant Dolbeault cohomology of a modification of the Lie algebra $G$.

Consider the same assumptions as in Theorem 5.2 and the DBA

$$B^{p,q}_\Gamma = \bigoplus_{a+c=p,b+d=q} \bigwedge^a\langle x_1,\dots,x_n\rangle \otimes \bigwedge^b\langle \bar x_1,\dots,\bar x_n\rangle \otimes \left\langle \beta_I\gamma_J y_I\wedge \bar y_J \middle| \begin{array}{c} |I|=c,\, |J|=d \\ (\beta_J\gamma_L)_{|\Gamma}=1 \end{array}\right\rangle$$

such that the inclusion $B^{*,*}_\Gamma \subset A^{*,*}(G/\Gamma)$ induces a cohomology isomorphism.

Let
$$\rho : G \to GL(\mathfrak{n}_{\mathbb{C}})$$
such that $\rho(g)Y_i = \beta_i Y_i$ and $\rho(g)\bar Y_i = \gamma_i \bar Y_i$.

**Lemma 6.1.** *We denote by* $\mathrm{Aut}^d(\mathfrak{n}_{\mathbb{C}})$ *the group of automorphisms of the complex Lie algebra* $\mathfrak{n}_{\mathbb{C}}$ *which are diagonalized by the basis* $Y_1,\dots,Y_m,\bar Y_1,\dots,\bar Y_m$. *Then for any* $g \in G$, *we have* $\rho(g) \in \mathrm{Aut}^d(\mathfrak{n}_{\mathbb{C}})$.

*Proof.* For $f \in \mathrm{Aut}^d(\mathfrak{n}_{\mathbb{C}})$, we have $f[Y_i,Y_j] = [fY_i, fY_j]$, $f[Y_i,\bar Y_j] = [fY_i, f\bar Y_j]$ and $f[\bar Y_i,\bar Y_j] = [f\bar Y_i, f\bar Y_j]$. Choosing the parameters $(t_1,\dots,t_m,s_1,\dots,s_m)$ such that $fY_i = t_iY_i$ and $f\bar Y_i = s_iY_i$, the algebraic group $\mathrm{Aut}^d(\mathfrak{n}_{\mathbb{C}})$ is then defined by equations of the form
$$t_1^{i_1}\cdots t_m^{i_m} s_1^{j_1}\cdots s_m^{j_m} = t_1^{k_1}\cdots t_m^{k_m} s_1^{l_1}\cdots s_m^{l_m}.$$
If the conditions
$$\alpha_1^{i_1}\dots\alpha_m^{i_m}\bar\alpha_1^{j_1}\dots\bar\alpha_m^{j_m} = \alpha_1^{k_1}\dots\alpha_m^{k_m}\bar\alpha_1^{l_1}\dots\bar\alpha_m^{l_m}$$
hold, thus by the definition of $\beta_i$ and $\gamma_i$ we get
$$\beta_1^{i_1}\dots\beta_m^{i_m}\gamma_1^{j_1}\dots\gamma_m^{j_m} = \beta_1^{k_1}\dots\beta_m^{k_m}\gamma_1^{l_1}\dots\gamma_m^{l_m}.$$
Hence, since by construction $\mathrm{Ad}_{sg} \in \mathrm{Aut}^d(\mathfrak{n}_{\mathbb{C}})$, we have $\rho(t) \in \mathrm{Aut}^d(\mathfrak{n}^C)$. □

We have that $T = \mathcal{A}(\rho(G))$ and thus by the previous lemma $T \subset \mathrm{Aut}^d(\mathfrak{n}_{\mathbb{C}})$. Let $S^\perp$ the connected subtorus of $T$ defined by
$$S^\perp = \mathcal{A}(\rho(\Gamma)).$$
We assume $S^\perp$ is connected. In case $S^\perp$ is not connected, we can take a finite dimensional subgroup $\tilde\Gamma \subset \Gamma$ such that $\mathcal{A}(\rho(\Gamma))$ is connected. We have the direct decomposition $T = S \times S^\perp$, where $S$ is a complement of $S^\perp$ in $T$. Consider the projection
$$\pi : G \to T \to S,$$

then

$$\pi : G \to (\tilde\beta_1, \dots, \tilde\beta_m, \tilde\gamma_1, \dots, \tilde\gamma_m).$$

Consider the vector spaces

$$\tilde{\mathfrak{g}}^{1,0} = \mathrm{span}\langle X_1, \dots X_n, \tilde\beta_1^{-1} Y_1, \dots, \tilde\beta_m^{-1} Y_m\rangle,$$
$$\tilde{\mathfrak{g}}^{0,1} = \mathrm{span}\langle \overline{X}_1, \dots \overline{X}_n, \tilde\gamma_1^{-1} \overline{Y}_1, \dots, \tilde\gamma_m^{-1} \overline{Y}_m\rangle,$$

with duals

$$(\tilde{\mathfrak{g}}^{1,0})^* = \mathrm{span} < x_1, \dots x_n, \tilde\beta_1 y_1, \dots, \tilde\beta_m y_m >,$$
$$(\tilde{\mathfrak{g}}^{0,1})^* = \mathrm{span} < \overline{x}_1, \dots \overline{x}_n, \tilde\gamma_1 \overline{y}_1, \dots, \tilde\gamma_m \overline{y}_m >,$$

and the subcomplex $\bigwedge(\tilde{\mathfrak{g}}^{1,0})^* \otimes (\tilde{\mathfrak{g}}^{0,1})^*$ of the Dolbeault complex $A^{*,*}(G/\Gamma)$. Then we can show the following:

**Lemma 6.2.** $B^{*,*}_\Gamma \subset \bigwedge(\tilde{\mathfrak{g}}^{1,0})^* \otimes (\tilde{\mathfrak{g}}^{0,1})^*$.

*Proof.* We regard $T \subset \mathrm{Aut}^d(\mathfrak{n}_\mathbb{C})$ as a subset of the complex diagonal matrices $D_{2m}(\mathbb{C})$. For

$$t_i \in \mathrm{Char}(D_{2m}(\mathbb{C})) = \{t_1^{i_1} \cdots t_n^{i_{2m}} s_1^{j_1} \dots s_m^{j_m} \,|\, \mathrm{diag}\,(t_1, \dots, t_n, s_1 \dots s_m) \in D_{2m}(\mathbb{C})\},$$

we denote $f_i = (t_i)_{|T}$, $f'_i = (s_i)_{|T}$ , $g_i = (t_i)_{|S} \circ p_S$, $g'_i = (s_i)_{|S} \circ p_S$, $h_i = (t_i)_{|S^\perp} p_{S^\perp}$ and $h'_i = (s_i)_{|S^\perp} \circ p_{S^\perp}$ where $p_S : T \to S$ and $p_{S^\perp} : T \to S^\perp$ are the projections. We have $f_i \circ \rho = \beta_i$, $f'_i \circ \rho = \gamma_i$, $g_i \circ \rho = \tilde\beta_i$, $g'_i \circ \rho = \tilde\gamma_i$ and $f_i = g_i h_i$ and $f'_i = g'_i h'_i$. Therefore $\rho(g)$ is diagonal, for every $g \in G$.

Suppose that $(\beta_J \gamma_L)\lfloor_\Gamma = 1$ for some $J, L \subseteq \{1, \dots, m\}$ and consider

$$f_J f'_L = g_J g'_L h_J h'_L.$$

Then, since $\rho(\Gamma) \subset S^\perp$, we have $(g_J g'_L)_{|\rho(\Gamma)} = 1$. Hence

$$1 = (\beta_J \gamma_L)\lfloor_\Gamma = (f_J f'_L)_{\rho(\Gamma)} = (h_J h'_L)_{\rho(\Gamma)}.$$

Since $S^\perp$ is the Zariski closure of $\rho(\Gamma)$, we have $h_J h'_L = 1$ and we get $f_J f'_L = g_J g'_L$. Hence if $(\beta_J \gamma_L)\lfloor_\Gamma = 1$, we obtain

$$\beta_J \gamma_L = \tilde\beta_J \tilde\gamma_L$$

and thus the lemma follows. □

By using the previous Lemma, Theorem 5.2, and the the inclusions $B^{*,*}_\Gamma \subset \bigwedge(\tilde{\mathfrak{g}}^{1,0})^* \otimes (\tilde{\mathfrak{g}}^{0,1})^* \subset A^{*,*}(G/\Gamma)$, we get the isomorphism

$$H^{*,*}_{\overline\partial}(G/\Gamma) \cong H^{*,*}_{\overline\partial}\left(\bigwedge(\tilde{\mathfrak{g}}^{1,0})^* \otimes (\tilde{\mathfrak{g}}^{0,1})^*\right).$$

Since $\tilde\beta_i$ and $\tilde\gamma_i$ are unitary, there exist holomorphic characters $\delta_i$ such that

$$\frac{\bar\delta_i}{\delta_i} = \tilde\beta_i \tilde\gamma_i.$$

Then we can prove the following

**Lemma 6.3.** *Let* $\rho' : G \to GL(\mathfrak{n}_\mathbb{C})$ *such that* $\rho'(g) X_i = \delta_i X_i$ *and* $\rho'(g) \bar X_i = \delta_i \bar X_i$. *Then we have* $\rho'(g) \in \mathrm{Aut}^d(\mathfrak{n}_\mathbb{C})$.

*Proof.* Consider the equations

$$t_1^{i_1} \cdots t_m^{i_m} s_1^{j_1} \cdots s_m^{j_m} = t_1^{k_1} \cdots t_m^{k_m} s_1^{l_1} \cdots s_m^{l_m}$$

defining the algebraic group $\mathrm{Aut}^d(\mathfrak{n}_\mathbb{C})$, as in the proof of Lemma 6.1. Suppose that the equation

$$\alpha_1^{i_1} \dots \alpha_m^{i_m} \bar\alpha_1^{j_1} \dots \bar\alpha_m^{j_m} = \alpha_1^{k_1} \dots \alpha_m^{k_m} \bar\alpha_1^{l_1} \dots \bar\alpha_m^{l_m}$$

holds. Then
$$\beta_1^{i_1}\dots\beta_m^{i_m}\gamma_1^{j_1}\dots\gamma_m^{j_m}=\beta_1^{k_1}\dots\beta_m^{k_m}\gamma_1^{l_1}\dots\gamma_m^{l_m}.$$
By the complex conjugation of first equation
$$\bar\alpha_1^{i_1}\dots\bar\alpha_m^{i_m}\alpha_1^{j_1}\dots\alpha_m^{j_m}=\bar\alpha_1^{k_1}\dots\bar\alpha_m^{k_m}\alpha_1^{l_1}\dots\alpha_m^{l_m}$$
we obtain
$$\gamma_1^{i_1}\dots\gamma_m^{i_m}\beta_1^{j_1}\dots\beta_m^{j_m}=\gamma_1^{k_1}\dots\gamma_m^{k_m}\beta_1^{l_1}\dots\beta_m^{l_m}.$$
Hence
$$\beta_1^{i_1}\dots\beta_m^{i_m}\gamma_1^{i_1}\dots\gamma_m^{i_m}\beta_1^{j_1}\dots\beta_m^{j_m}\gamma_1^{j_1}\dots\gamma_m^{j_m}\quad=\quad\beta_1^{k_1}\dots\beta_m^{k_m}\gamma_1^{k_1}\dots\gamma_m^{k_m}\beta_1^{k_1}\dots\beta_m^{l_m}\gamma_1^{l_1}\dots\gamma_m^{l_m}.$$
Since $S$ is a sub-torus of $T=\mathcal{A}(\rho(G))$, we have
$$\tilde\beta_1^{i_1}\dots\tilde\beta_m^{i_m}\tilde\gamma_1^{i_1}\dots\tilde\gamma_m^{i_m}\tilde\beta_1^{j_1}\dots\tilde\beta_m^{j_m}\tilde\gamma_1^{j_1}\dots\tilde\gamma_m^{j_m}\quad=\quad\tilde\beta_1^{k_1}\dots\tilde\beta_m^{k_m}\tilde\gamma_1^{k_1}\dots\tilde\gamma_m^{k_m}\tilde\beta_1^{k_1}\dots\tilde\beta_m^{l_m}\tilde\gamma_1^{l_1}\dots\tilde\gamma_m^{l_m}.$$
Thus
$$\frac{\bar\delta_1^{i_1}}{\delta_1^{i_1}}\cdots\frac{\bar\delta_m^{i_m}}{\delta_m^{i_m}}\frac{\bar\delta_1^{j_1}}{\delta_1^{j_1}}\cdots\frac{\bar\delta_1^{j_m}}{\delta_1^{j_m}}=\frac{\bar\delta_1^{k_1}}{\delta_1^{k_1}}\cdots\frac{\bar\delta_m^{k_m}}{\delta_m^{k_m}}\frac{\bar\delta_1^{l_1}}{\delta_1^{l_1}}\cdots\frac{\bar\delta_1^{l_m}}{\delta_1^{l_m}}.$$
Since $\delta_i$ is holomorphic, the following relation holds
$$\delta_1^{i_1}\dots\delta_m^{i_m}\delta_1^{j_1}\dots\delta_m^{j_m}=\delta_1^{k_1}\dots\delta_m^{k_m}\delta_1^{l_1}\dots\delta_m^{l_m}.$$
Hence $\rho'(g)\in\mathrm{Aut}^d(\mathfrak{n}_{\mathbb{C}})$. □

Take
$$\breve{\mathfrak{g}}^{1,0}=\mathrm{span}<X_1,\dots X_n,\tilde\beta_1^{-1}\delta_1^{-1}Y_1,\dots,\tilde\beta_m^{-1}\delta_m^{-1}Y_m>,$$
$$\breve{\mathfrak{g}}^{0,1}=\mathrm{span}<\overline{X}_1,\dots\overline{X}_n,\tilde\gamma_1^{-1}\delta_1^{-1}\overline{Y}_1,\dots,\tilde\gamma_m^{-1}\delta_m^{-1}\overline{Y}_m>,$$
Then $\overline{\tilde\beta_i\delta_i}=\tilde\gamma_i\delta_i$ yields
$$\overline{\breve{\mathfrak{g}}^{1,0}}=\breve{\mathfrak{g}}^{0,1}.$$
Hence the complex Lie algebra $\breve{\mathfrak{g}}^{1,0}\oplus\breve{\mathfrak{g}}^{0,1}$ has a real form $\breve{\mathfrak{g}}$ endowed with a complex structure $\breve J$. Consider the cochain complex
$$\bigwedge(\breve{\mathfrak{g}}^{1,0})^*\otimes(\breve{\mathfrak{g}}^{0,1})^*.$$
Since $\delta_i$ is holomorphic, we have a cochain complex isomorphism
$$\bigwedge(\breve{\mathfrak{g}}^{1,0})^*\otimes(\breve{\mathfrak{g}}^{0,1})^*\cong\bigwedge(\tilde{\mathfrak{g}}^{1,0})^*\otimes(\tilde{\mathfrak{g}}^{0,1})^*.$$
Hence $H^{*,*}_{\bar\partial}(G/\Gamma)\cong H^{*,*}_{\bar\partial}(\breve{\mathfrak{g}},\breve J)$ and we have shown the following

**Theorem 6.4.** *Let $G$ be a simply connected solvable Lie group with a left-invariant complex structure $J$, a lattice $\Gamma$ and the nilradical $N$. Let $\mathfrak{g}$ be the Lie algebra of $G$ and $\mathfrak{n}$ be the Lie algebra of $N$. Suppose $J\mathfrak{n}\subseteq\mathfrak{n}$ and hence $J$ induces complex structures on the nilmanifold $N/\Gamma\cap N$ and on the torus $G/N\Gamma$. Assume furthermore that $\mathrm{Ad}_{sg}\circ J=J\circ\mathrm{Ad}_{sg}$ for any $g\in G$, that the Mostow fibration*
$$N/\Gamma\cap N\to G/\Gamma\to G/N\Gamma$$
*is holomorphic and $H^{*,*}_{\bar\partial}(N/\Gamma\cap N)\cong H^{*,*}_{\bar\partial}(\mathfrak{n})$.*

*Then there exists a finite index subgroup $\tilde\Gamma$ of $\Gamma$ and a real Lie algebra $\breve{\mathfrak{g}}$ with a complex structure $\breve J$ such that*
$$H^{*,*}_{\bar\partial}(G/\tilde\Gamma)\cong H^{*,*}_{\bar\partial}(\breve{\mathfrak{g}},\breve J).$$

**Corollary 6.5.** *In the same assumptions as in Theorem 6.4, we suppose furthermore that $\mathfrak{g}$ is completely solvable. Then there exists a finite index subgroup $\tilde{\Gamma}$ of $\Gamma$ and a completely solvable real Lie algebra $\breve{\mathfrak{g}}$ with a complex structure $\breve{J}$ such that*

$$H^{*,*}_{\overline{\partial}}(G/\tilde{\Gamma}) \cong H^{*,*}_{\overline{\partial}}(\breve{\mathfrak{g}}, \breve{J}).$$

*Proof.* Since $\alpha_i = \bar{\alpha}_i$, we have $\beta_i = \gamma_i$ and hence $\tilde{\beta}_i = \tilde{\gamma}_i$. We have $\delta_i = \tilde{\alpha}_i \overline{\tilde{\beta}_i}$ for some real character $\tilde{\alpha}_i$ and thus

$$\breve{\mathfrak{g}}^{1,0} = \text{span} < X_1, \dots X_n, \tilde{\alpha}_1^{-1} Y_1, \dots, \tilde{\alpha}_m^{-1} Y_m >,$$
$$\breve{\mathfrak{g}}^{0,1} = \text{span} < \overline{X}_1, \dots \overline{X}_n, \tilde{\alpha}_1^{-1} Y_1, \dots, \tilde{\alpha}_m^{-1} Y_m > .$$

□

In a similar way we can extend the previous result to solvable Lie groups $G$ endowed with an abelian complex structure. Recall that a complex structure $J$ on $\mathfrak{g}$ is called *abelian* if $\mathfrak{g}^{1,0}$ is abelian. This means

$$[JX, JY] = [X, Y], \quad \forall X, Y \in \mathfrak{g}\,,$$

or equivalently

$$d(\mathfrak{g}^{1,0})^* \subset (\mathfrak{g}^{1,1})^*\,.$$

Note that since $J$ is abelian we always have [6, 8]

$$H^{*,*}_{\overline{\partial}}(\mathfrak{n}) \cong H^{*,*}_{\overline{\partial}}(N/\Gamma_N).$$

**Theorem 6.6.** *In the same assumptions as in Theorem 6.4, we suppose furthermore that $J$ is abelian. Then there exists a finite index subgroup $\tilde{\Gamma}$ of $\Gamma$ and a solvable Lie algebra $\breve{\mathfrak{g}}$ endowed with an abelian complex structure $\breve{J}$ such that*

$$H^{*,*}_{\overline{\partial}}(G/\tilde{\Gamma}) \cong H^*(\breve{\mathfrak{g}}^{1,0}) \otimes \wedge^* \breve{\mathfrak{g}}^{0,1}\,.$$

*Proof.* Since $J$ is abelian, we have

$$\mathrm{ad}_X(JY) = -\mathrm{ad}_{JX} Y, \quad \forall X, Y \in \mathfrak{g}.$$

If $Z \in \mathfrak{g}^{1,0}$, i.e. if $JZ = \sqrt{-1}Z$, we get

$$(\mathrm{ad}_X + \sqrt{-1} J \mathrm{ad}_X)(Z) = 0, \quad \forall X \in \mathfrak{g},$$

and so in particular that $\mathrm{ad}_s$ vanishes on $\mathfrak{g}^{1,0}$.

If we consider the decomposition $\mathfrak{n}_{\mathbb{C}} = \mathfrak{n}^{1,0} \oplus \mathfrak{n}^{0,1}$, then by our assumptions there exists a basis $Y_1, \dots, Y_m$ of $\mathfrak{n}^{1,0}$ such that the action $\mathrm{Ad}_s$ on $\mathfrak{n}^{1,0}$ is represented by

$$\mathrm{Ad}_{sg} = \mathrm{diag}\,(\alpha_1(g), \dots, \alpha_m(g)),$$

where $\alpha_i$ is anti-holomorphic for every $i$. By using Lemma 2.2 in [15], for every $\alpha_i$ there exist a unique unitary character $\beta_i$ such that $\alpha_i \beta_i^{-1}$ is holomorphic and a unique unitary character $\gamma_i$ such that $\overline{\alpha}_i \gamma_i^{-1}$ is holomorphic. In this case $\beta_i = \frac{\alpha_i}{\overline{\alpha}_i}$ and $\gamma_i$ is trivial.

Let

$$\rho : G \to GL(\mathfrak{n}_{\mathbb{C}})$$

be such that $\rho(g) X_i = \beta_i X_i$ and $\rho(g) \overline{X}_i = \gamma_i \overline{X}_i$. As in Lemma 6.1 one can show tthat $\rho(g) \in \mathrm{Aut}^d(\mathfrak{n}_{\mathbb{C}})$.

Then consider as before

$$\pi : G \to (\tilde{\beta}_1, \dots, \tilde{\beta}_m, \tilde{\gamma}_1, \dots, \tilde{\gamma}_m),$$

with $\tilde{\gamma}_i$ trivial.

Since $\tilde{\beta}_i$ and $\tilde{\gamma}_i$ are unitary, we have holomorphic characters $\delta_i$ such that

$$\frac{\overline{\delta}_i}{\delta_i} = \tilde{\beta}_i .$$

We can show that the complex structure $\breve{J}$ is abelian since

$$\breve{\mathfrak{g}}^{1,0} = \mathrm{span} < X_1, \dots X_n, \overline{\delta}_1^{-1} Y_1, \dots, \overline{\delta}_m^{-1} Y_m >$$

and $\overline{\delta}_i^{-1}$ are anti-holomorphic.

□

**Example 6.7.** We can apply Theorem 6.6 to the Lie group $G = \mathbb{C} \ltimes_\phi \mathbb{C}^2$ with

$$\phi(z_1) = \begin{pmatrix} e^{\overline{z}_1} & 0 \\ 0 & e^{-\overline{z}_1} \end{pmatrix} .$$

Consider on $G$ the abelian complex structure defined by

$$\mathfrak{g}^{1,0} = \mathrm{span} \left\langle \frac{\partial}{\partial z_1}, e^{\bar{z}_1} \frac{\partial}{\partial z_2}, e^{-\bar{z}_1} \frac{\partial}{\partial z_3} \right\rangle .$$

Any lattice $\Gamma$ in $G$ is of the form $(\mathbb{Z} + 2\pi\sqrt{-1}\mathbb{Z}) \ltimes \Gamma'$ with $\Gamma'$ lattice in $N = \mathbb{C}^2$. In this case we have $\delta_i = \overline{\alpha}_i$ and $\tilde{\alpha}_i = \alpha_i$. Since

$$\alpha_1 = e^{\overline{z}_1}, \quad \alpha_2 = e^{-\overline{z}_1}$$

we obtain that

$$\breve{\mathfrak{g}}^{1,0} = \mathrm{span} < X_1, Y_1, Y_2 >= \mathrm{span} \left\langle \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3} \right\rangle .$$

Therefore the modification $(\breve{\mathfrak{g}}, \breve{J})$ is the abelian Lie algebra $\mathbb{C}^3$.

In the case of a *complex parallelizable compact solvmanifold* we can show the following

**Corollary 6.8.** *Let $G$ be a simply connected complex solvable Lie group with a lattice $\Gamma$. Then there exists a finite index subgroup $\tilde{\Gamma}$ of $\Gamma$ and a complex Lie algebra $\breve{\mathfrak{g}}$ such that*

$$H^{*,*}_{\overline{\partial}}(G/\tilde{\Gamma}) \cong \wedge^* \breve{\mathfrak{g}}^{1,0} \otimes H^*(\breve{\mathfrak{g}}^{0,1}) .$$

*Proof.* In case $\mathfrak{g}$ is a complex solvable Lie algebra, $\alpha_i$ is holomorphic and so $\beta_i$ and $\tilde{\beta}_i$ are trivial. Hence we have $\bar{\delta}_i = \delta_i \tilde{\gamma}_i$ and

$$\breve{\mathfrak{g}}^{1,0} = \mathrm{span} < X_1, \dots X_n, \delta_1^{-1} Y_1, \dots, \delta_m^{-1} Y_m >,$$
$$\breve{\mathfrak{g}}^{0,1} = \mathrm{span} < \overline{X}_1, \dots \overline{X}_n, \bar{\delta}_1^{-1} \overline{Y}_1, \dots, \bar{\delta}_m^{-1} \overline{Y}_m >,$$

□

**Remark 6.9.** In general, it is not true that there exists a simply connected solvable Lie group $\breve{G}$ with the Lie algebra $\breve{\mathfrak{g}}$ containing $\tilde{\Gamma}$ as a lattice. For example, let $G = \mathbb{C} \ltimes_\phi \mathbb{C}^2$ such that

$$\phi(x + \sqrt{-1}y) = \begin{pmatrix} e^{x+\sqrt{-1}y} & 0 \\ 0 & e^{-x-\sqrt{-1}y} \end{pmatrix}$$

with a lattice $\Gamma = (a\mathbb{Z} + 2\pi\sqrt{-1}) \ltimes \Gamma''$. Then we have $H^{*,*}(G/\Gamma) \cong \bigwedge \mathbb{C}^3$ and hence we get $\breve{\mathfrak{g}} \cong \mathbb{C}^3$. But any lattice in $G$ cannot be embedded in $\mathbb{C}^3$.

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